[illegible]

Exercise 1. First, observe that for $x \in \mathbb{R}$,

$$\cos(2x) + 3\sin(x) - 3 = -2\sin^2(x) + 3\sin(x) - 2 = -(2\sin(x) - 1)(\sin(x) - 2).$$

We now solve the equation in $\mathbb{R}$. Let $x \in \mathbb{R}$:

$$\begin{aligned} \cos(2x) + 3\sin(x) - 3 = 0 &\iff \sin(x) = \frac{1}{2} \text{ or } \sin(x) = 2 \\ &\iff \sin(x) = \sin\left(\frac{\pi}{6}\right) \qquad \text{since it's impossible to have } \sin(x) = 2 \\ &\iff \exists k \in \mathbb{Z},\ \left(x = \frac{\pi}{6} + 2k\pi \text{ or } x = \frac{5\pi}{6} + 2k\pi\right). \end{aligned}$$

We now select the solutions that lie in $[-2\pi, 2\pi]$.

- For the solutions of the form $\pi/6 + 2k\pi$: let $k \in \mathbb{Z}$:

$$\begin{aligned} \frac{\pi}{6} + 2k\pi \in [-2\pi, 2\pi] &\iff -2\pi \le \frac{\pi}{6} + 2k\pi \le 2\pi \\ &\iff -2 \le \frac{1}{6} + 2k \le 2 \\ &\iff -\frac{13}{6} \le 2k \le \frac{11}{6} \\ &\iff -\frac{13}{12} \le k \le \frac{11}{12} \\ &\iff k \in \{-1, 0\}. \end{aligned}$$

  This corresponds to the solutions

$$-\frac{11\pi}{6}, \frac{\pi}{6}.$$

- For the solutions of the form $5\pi/6 + 2k\pi$: let $k \in \mathbb{Z}$:

$$\begin{aligned} \frac{5\pi}{6} + 2k\pi \in [-2\pi, 2\pi] &\iff -2\pi \le \frac{5\pi}{6} + 2k\pi \le 2\pi \\ &\iff -2 \le \frac{5}{6} + 2k \le 2 \\ &\iff -\frac{17}{6} \le 2k \le \frac{7}{6} \\ &\iff -\frac{17}{12} \le k \le \frac{7}{12} \\ &\iff k \in \{-1, 0\}. \end{aligned}$$

  This corresponds to the solutions

$$-\frac{7\pi}{6}, \frac{5\pi}{6}.$$

Conclusion, the set of solutions is

$$\left\{-\frac{11\pi}{6}, \frac{\pi}{6}, -\frac{7\pi}{6}, \frac{5\pi}{6}\right\}.$$

Exercise 2.

1. See Figure 1.

2.

$$f([0,2]) = [0,2), \qquad f([0,1)) = (0,1],$$

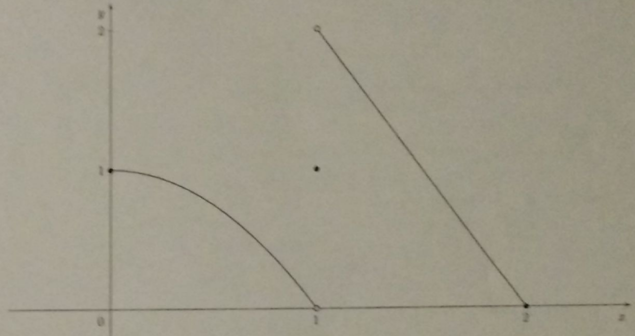

Figure 1 – Graph of the function f of Exercise 2.

$$f((0,1]) = (0,1], \qquad f([1/2, 5/4]) = \left(0, \frac{3}{4}\right] \cup \{1\} \cup \left[\frac{3}{2}, 2\right),$$

$$f^{(-1)}([0,2]) = [0,2], \qquad f^{(-1)}([0,1]) = [0,1] \cup \left[\frac{3}{2}, 2\right],$$

$$f^{(-1)}(\{1\}) = \left\{0, 1, \frac{3}{2}\right\}, \qquad f^{(-1)}(\{2\}) = \emptyset.$$

3. Let $x, y \in (1, 2]$ such that $x < y$. Then: $-2x > -2y$, hence $4 - 2x > 4 - 2y$, hence $f(x) > f(y)$.

4. The function f is not injective since, e.g., $f(0) = f(1) = 1$. The function f is not surjective since there are no elements $x \in [0,2]$ such that $f(x) = 2$, as shown by $f^{(-1)}(\{2\}) = \emptyset$. The function f is not bijective since f is not injective (or not surjective).

Exercise 3.

1. Let $x \in \mathbb{R}$ such that $\cos(x) \neq 0$. Then:

$$1 - \frac{e^{ix}}{\cos(x)} = \frac{\cos(x) - (\cos(x) + i\sin(x))}{\cos(x)} = \frac{-i\sin(x)}{\cos(x)} = -i\tan(x).$$

2. This expression is well-defined provided $\cos(x) \neq 0$, i.e., for $x \in (-\pi, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, \pi)$.

3. Let $x \in (-\pi, -\pi/2) \cup (-\pi/2, 0) \cup (0, \pi/2) \cup (\pi/2, \pi)$. Then:

$$\sum_{k=0}^{n} \frac{\cos(kx)}{\cos^k(x)} = \sum_{k=0}^{n} \frac{\operatorname{Re}(e^{ikx})}{\cos^k(x)} = \operatorname{Re}\left(\sum_{k=0}^{n} \left(\frac{e^{ix}}{\cos(x)}\right)^k\right).$$

   Notice that from our assumption on x, we have

$$\frac{e^{ix}}{\cos(x)} \neq 1,$$

   hence, from the formula for the sum of geometric terms,

$$\sum_{k=0}^{n} \frac{\cos(kx)}{\cos^k(x)} = \operatorname{Re}\left(\frac{1 - \frac{e^{i(n+1)x}}{\cos^{n+1}(x)}}{1 - \frac{e^{ix}}{\cos(x)}}\right) = \operatorname{Re}\left(\frac{1 - \frac{e^{i(n+1)x}}{\cos^{n+1}(x)}}{-i\tan(x)}\right) \qquad \text{by Question 1}$$

$$\begin{aligned}
&= \operatorname{Re}\left(\frac{\cos^{n+1}(x) - e^{i(n+1)x}}{-i\cos^{n+1}(x)\tan(x)}\right) = \operatorname{Re}\left(\frac{i\cos^{n+1}(x) - ie^{i(n+1)x}}{\cos^{n}(x)\sin(x)}\right)\\
&= \operatorname{Re}\left(\frac{i\cos^{n+1}(x) - i\cos((n+1)x) + \sin((n+1)x)}{\cos^{n}(x)\sin(x)}\right)\\
&= \frac{\sin((n+1)x)}{\cos^{n}(x)\sin(x)}.
\end{aligned}$$

Exercise 4.

1. a) Let P be a polynomial function with real coefficients of degree 3, say

$$\forall x \in \mathbb{R},\ P(x) = ax^3 + bx^2 + cx + d,$$

with $a, b, c, d \in \mathbb{R}$. Then, for $x \in \mathbb{R}$,

$$\begin{aligned}
P(x+1) - P(x) &= a(x+1)^3 + b(x+1)^2 + c(x+1) + d - ax^3 - bx^2 - cx - d\\
&= a(x^3 + 3x^2 + 3x + 1) + b(x^2 + 2x + 1) + c(x+1) - ax^3 - bx^2 - cx\\
&= a(3x^2 + 3x + 1) + b(2x + 1) + c\\
&= 3ax^2 + (3a + 2b)x + a + b + c.
\end{aligned}$$

Hence, by identification, we want:

$$\begin{cases} 3a & = 1\\ 3a + 2b & = 0\\ a + b + c & = 0 \end{cases} \iff \begin{cases} a = 1/3\\ b = -1/2\\ c = 1/6. \end{cases}$$

Notice that we have no conditions on d; we might as well take $d = 0$. Hence, the polynomial function

$$\begin{aligned} P : \mathbb{R} &\longrightarrow \mathbb{R}\\ x &\longmapsto \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6} \end{aligned}$$

will do.

b)

$$\sum_{k=0}^{n} (u_{k+1} - u_k) = u_{n+1} - u_0.$$

c) Let $n \in \mathbb{N}$. Then:

$$\begin{aligned}
\sum_{k=0}^{n} k^2 &= \sum_{k=0}^{n} \big(P(k+1) - P(k)\big)\\
&= P(n+1) - P(0)\\
&= \frac{(n+1)^3}{3} - \frac{(n+1)^2}{2} + \frac{n+1}{6}\\
&= \frac{2(n+1)^2 - 3(n+1) + 1}{6}(n+1)\\
&= \frac{2n^2 + n}{6}(n+1)\\
&= \frac{(2n+1)n(n+1)}{6}.
\end{aligned}$$

2. a) Let $n \in \mathbb{N}^*$:

$$\begin{aligned}
u_{n+1} - u_n &= \frac{n(2n+1)}{(n+1)^2} - \frac{(n-1)(2n-1)}{n^2} = \frac{n^3(2n+1) - (n-1)(2n-1)(n+1)^2}{(n+1)^2 n^2}\\
&= \frac{2n^4 + n^3 - (2n^4 + n^3 - 3n^2 - n + 1)}{(n+1)^2 n^2} = \frac{3n^2 + n - 1}{(n+1)^2 n^2}\\
&\geq \frac{3n^2}{(n+1)^2 n^2} \geq \frac{3}{(n+1)^2 n^2} > 0,
\end{aligned}$$

hence the sequence $(u_n)_{n\geq 1}$ is increasing.

b) Let $n \in \mathbb{N}^*$. Then:

$$0 \leq 1 - \frac{1}{n} < 1 \quad \text{and} \quad 0 \leq 2 - \frac{1}{n} < 2, \quad \text{hence} \quad \left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) < 2,$$

hence $u_n < 2$, hence 2 is an upper bound of the sequence $(u_n)_{n\geq 1}$.

c) Let $M \in (1, 2)$ and $n \in \mathbb{N}^*$ with

$$n > \frac{2}{3 - \sqrt{4M+1}}.$$

Then:

$$\frac{1}{n} < \frac{3 - \sqrt{4M+1}}{2},$$

hence

$$1 - \frac{1}{n} > \frac{-1 + \sqrt{4M+1}}{2} > 0 \quad \text{and} \quad 2 - \frac{1}{n} > \frac{1 + \sqrt{4M+1}}{2} > 0.$$

Hence

$$u_n = \left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) > \left(\frac{-1 + \sqrt{4M+1}}{2}\right)\left(\frac{1 + \sqrt{4M+1}}{2}\right) = \frac{4M}{4} = M.$$

Hence, if $M < 2$ then M is not an upper bound of the sequence $(u_n)_{n\geq 1}$. Since 2 is an upper bound of the sequence $(u_n)_{n\geq 1}$ we conclude that 2 is the least upper bound of the sequence $(u_n)_{n\geq 1}$.

3. a) On Figure 3, the small dashed arc of circle has a radius equal to t_1 and an angle equal to $t_2 - t_1$, hence it encloses a portion of disk of surface area $t_1^2(t_2 - t_1)/2$. The large dashed arc of circle has a radius equal to t_2 and an angle equal to $t_2 - t_1$, hence it encloses a portion of disk of surface area $t_2^2(t_2 - t_1)/2$. Now, the gray surface clearly contains the small portion of disk, and is clearly included in the large portion of disk, hence the inequality.

b) For $n \geq 1$, cutting the surface $\mathscr{S}$ into n parts (each with angle π/n) yields

$$A = \sum_{k=0}^{n-1} A_{k\pi/n,(k+1)\pi/n},$$

hence, from the previous inequality:

$$\sum_{k=0}^{n-1} \left(\frac{k\pi}{n}\right)^2 \frac{\pi}{n} \leq A \leq \sum_{k=0}^{n-1} \left(\frac{(k+1)\pi}{n}\right)^2 \frac{\pi}{n},$$

hence the result.

c) From Question 1c, for all $n \in \mathbb{N}^*$,

$$\sum_{k=0}^{n-1} \frac{k^2\pi^3}{2n^3} = \frac{\pi^3}{2n^3}\sum_{k=0}^{n-1} k^2 = \frac{\pi^3}{2n^3}\frac{(n-1)n(2n-1)}{6} = \frac{\pi^3(n-1)(2n-1)}{12n^2} = \frac{\pi^3}{12}u_n,$$

hence the result.

4. a) Since $12A/\pi^3$ is an upper bound of the sequence $(u_n)_{n\geq 1}$, $12A/\pi^3$ is non-less than the least upper bound of the sequence $(u_n)_{n\geq 1}$, namely 2, hence

$$2 \leq \frac{12A}{\pi^3},$$

hence

$$\frac{\pi^3}{6} \leq A.$$

b) We would define the sequence $(v_n)_{n\geq 1}$ as

$$\forall n \geq 1,\ v_n = \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

and show that the sequence $(v_n)_{n\geq 1}$ is decreasing and that 2 is the greatest lower bound of the sequence $(v_n)_{n\geq 1}$. We'll then conclude that

$$A \leq \frac{\pi^3}{6}$$

and hence that

$$A = \frac{\pi^3}{6}.$$