INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #1 — 1h30

October 23, 2015

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Find all the numbers $x \in [-2\pi, 2\pi]$ such that:

$$\cos(2x) + 5\sin(x) - 3 = 0.$$

Exercise 2. Let f be the function[1] defined by

$$f : [0,2] \longrightarrow [0,2], \quad x \longmapsto \begin{cases} 1 - x^2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \\ 4 - 2x & \text{if } 1 < x \le 2. \end{cases}$$

1. Sketch the graph of f.
2. Determine (without any justifications) the following sets:

$$f\big([0,2]\big), \quad f\big([0,1)\big), \quad f\big((0,1]\big), \quad f\big([1/2,5/4]\big),$$
$$f^{[-1]}\big([0,2]\big), \quad f^{[-1]}\big([0,1]\big), \quad f^{[-1]}\big(\{1\}\big), \quad f^{[-1]}\big(\{2\}\big).$$

3. Show that the function f is decreasing on $(1,2]$.
4. Is the function f injective? surjective? bijective? (justify your answers).

Exercise 3. Let $n \in \mathbb{N}^*$.

1. Let $x \in \mathbb{R}$ such that $\cos(x) \neq 0$. Show that

$$1 - \frac{e^{ix}}{\cos(x)} = -i\tan(x).$$

2. For which values $x \in (-\pi, \pi)$ is the expression

$$\sum_{k=0}^{n} \frac{\cos(kx)}{\cos^k(x)}$$

well-defined?

3. Let x be such a value, and assume moreover that $x \neq 0$. Show that

$$\sum_{k=0}^{n} \frac{\cos(kx)}{\cos^k(x)} = \frac{\sin\big((n+1)x\big)}{\sin(x)\cos^n(x)}.$$

Exercise 4. The goal of this exercise is to compute the surface area of the gray portion shown on Figure 1. questions 1 and 2 are independent from each other, but the results of these questions are used in questions 3 and 4.

1. a) Find a polynomial function P with real coefficients of degree 3 such that

$$\forall x \in \mathbb{R},\ P(x+1) - P(x) = x^2.$$

   b) If $(u_n)_{n \ge 0}$ is a sequence of real or complex numbers, and $n \in \mathbb{N}$, recall (without any justifications) the value of the following sum:

$$\sum_{k=0}^{n} (u_{k+1} - u_k).$$

[1] You don't need to check that the function f is well-defined: we have checked that for you.

c) Deduce that

$$\forall n \in \mathbb{N},\ \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Note: you will get partial credit if you weren't able to do Question 1a but you can solve Question 1c by induction.

2. We define the sequence $(u_n)_{n\geq 1}$ as

$$\forall n \in \mathbb{N}^*,\ u_n = \left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right).$$

a) Show that the sequence $(u_n)_{n\geq 1}$ is increasing.

b) Show that 2 is an upper bound of the sequence $(u_n)_{n\geq 1}$.

c) We now show that 2 is the least upper bound of the sequence $(u_n)_{n\geq 1}$: let $M \in (1,2)$. Clearly, the number

$$2\left(3 - \sqrt{4M-1}\right)^{-1}$$

is well-defined and positive. Show that if $n \in \mathbb{N}^*$ is such that $n > 2\left(3 - \sqrt{4M-1}\right)^{-1}$ then $u_n > M$. Conclude.

3. We consider the curve $\mathscr{C}$ obtained by the locus of the following points:

$$\big(t\cos(t), t\sin(t)\big),\ t \in [0,\pi]$$

and the surface $\mathscr{S}$ enclosed by this curve and the x-axis (see Figure 1). The surface area of $\mathscr{S}$ is denoted by A.

a) Let t_1, t_2 be numbers such that $0 \leq t_1 < t_2 \leq \pi$, and consider the corresponding points M_1 and M_2 on the curve:

$$M_1\big(t_1\cos(t_1), t_1\sin(t_1)\big), \qquad M_2\big(t_2\cos(t_2), t_2\sin(t_2)\big).$$

We denote by A_{t_1,t_2} the surface area enclosed by the line segment $[OM_1]$, the portion of the curve $\mathscr{C}$ between M_1 and M_2, and the line segment $[M_2O]$ (see Figure 2). Show that

$$\frac{t_1^2(t_2-t_1)}{2} \leq A_{t_1,t_2} \leq \frac{t_2^2(t_2-t_1)}{2}.$$

b) Let $n \in \mathbb{N}^*$ and split the surface $\mathscr{S}$ into n parts, as shown on Figure 3 (the angle between two consecutive line segments is π/n; Figure 3 shows the special case $n = 15$). Explain why

$$\sum_{k=0}^{n-1} \frac{k^2\pi^3}{2n^3} \leq A \leq \sum_{k=0}^{n-1} \frac{(k+1)^2\pi^3}{2n^3}.$$

Figure 4 illustrates this inequality in the special case $n = 15$.

c) Deduce that

$$\forall n \in \mathbb{N}^*,\ \frac{\pi^3}{12n^2}(n-1)(2n-1) \leq A \leq \frac{\pi^3}{12n^2}(n+1)(2n+1).$$

4. The goal of this question is to show that

$$A = \frac{\pi^3}{6}.$$

Note that, from the left inequality obtained in Question 3c, we have

$$\forall n \in \mathbb{N}^*,\ \frac{\pi^3}{12}u_n \leq A,$$

where $(u_n)_{n\geq 1}$ is the sequence defined in Question 2.

a) Use the result of Question 2 to show that

$$\frac{\pi^3}{6} \leq A.$$

b) Briefly explain what auxiliary sequence $(v_n)_{n\geq 1}$ you would use, and what results you would show to obtain the inequality

$$A \leq \frac{\pi^3}{6}$$

and be able to conclude that $A = \pi^3/6$.

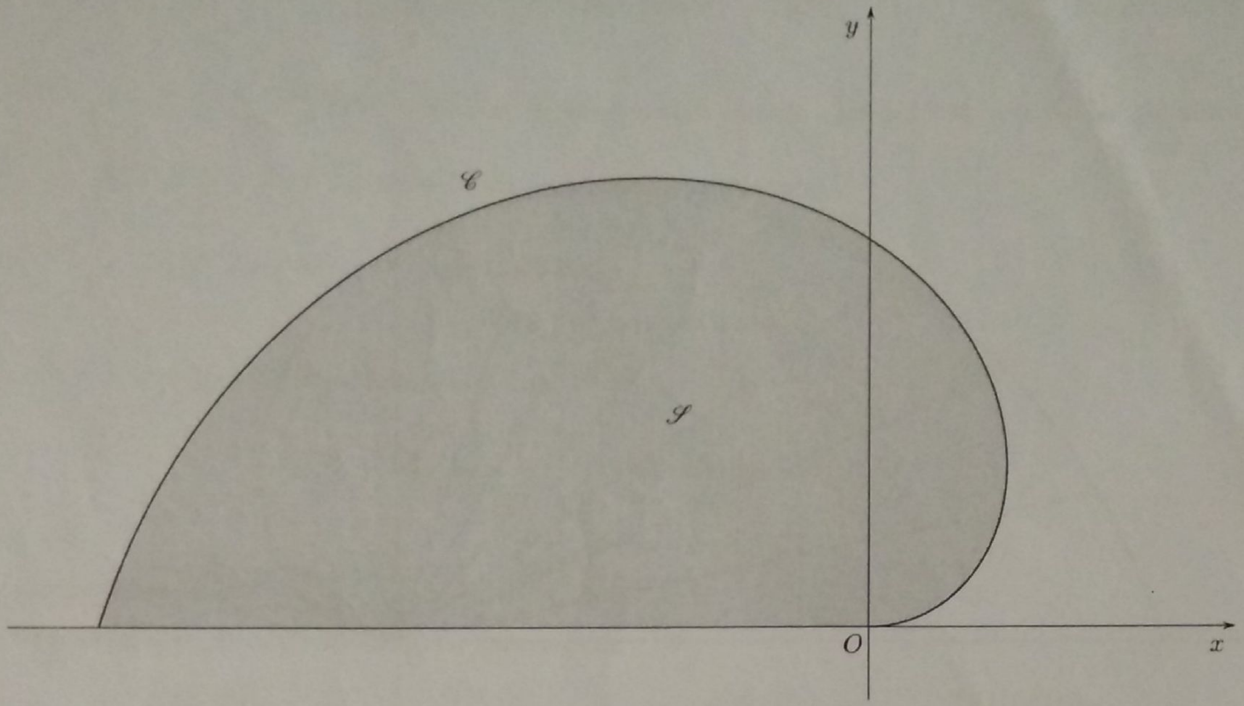

Figure 1 – Curve $\mathscr{C}$ and surface $\mathscr{S}$ of Exercise 4. The surface area of the gray part is denoted by A

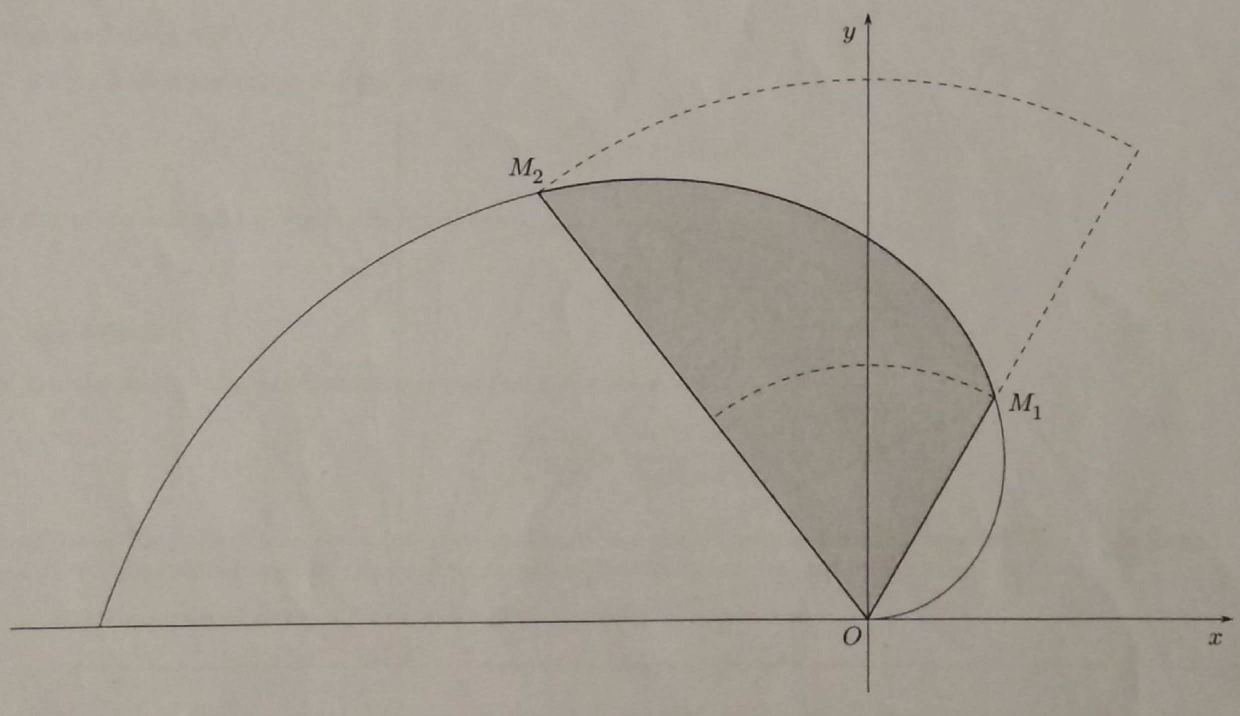

Figure 2 – Curve $\mathscr{C}$, points $M_1\big(t_1\cos(t_1), t_1\sin(t_1)\big)$ and $M_2\big(t_2\cos(t_2), t_2\sin(t_2)\big)$; the surface area of the gray portion is denoted by A_{t_1,t_2}. The dashed arcs are arcs of circles (to help you obtain the lower bound and upper bound in Question 3a)

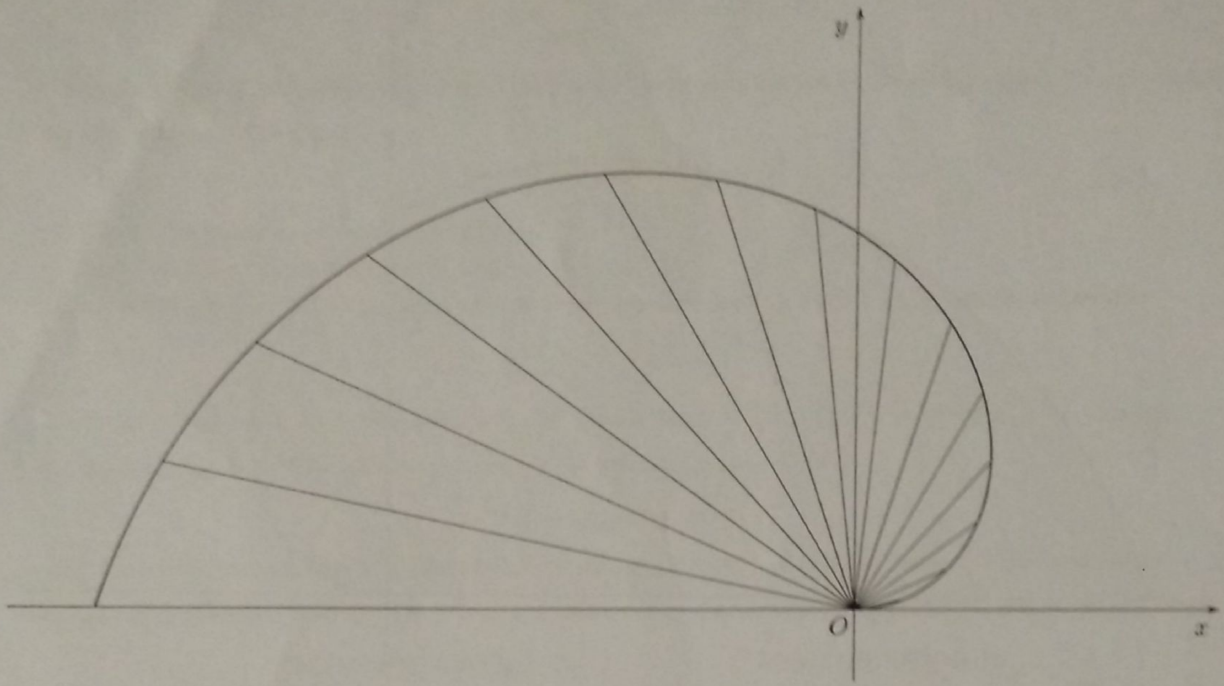

Figure 3 – Splitting of the surface $\mathscr{S}$ into n portions (shown here with $n = 15$)

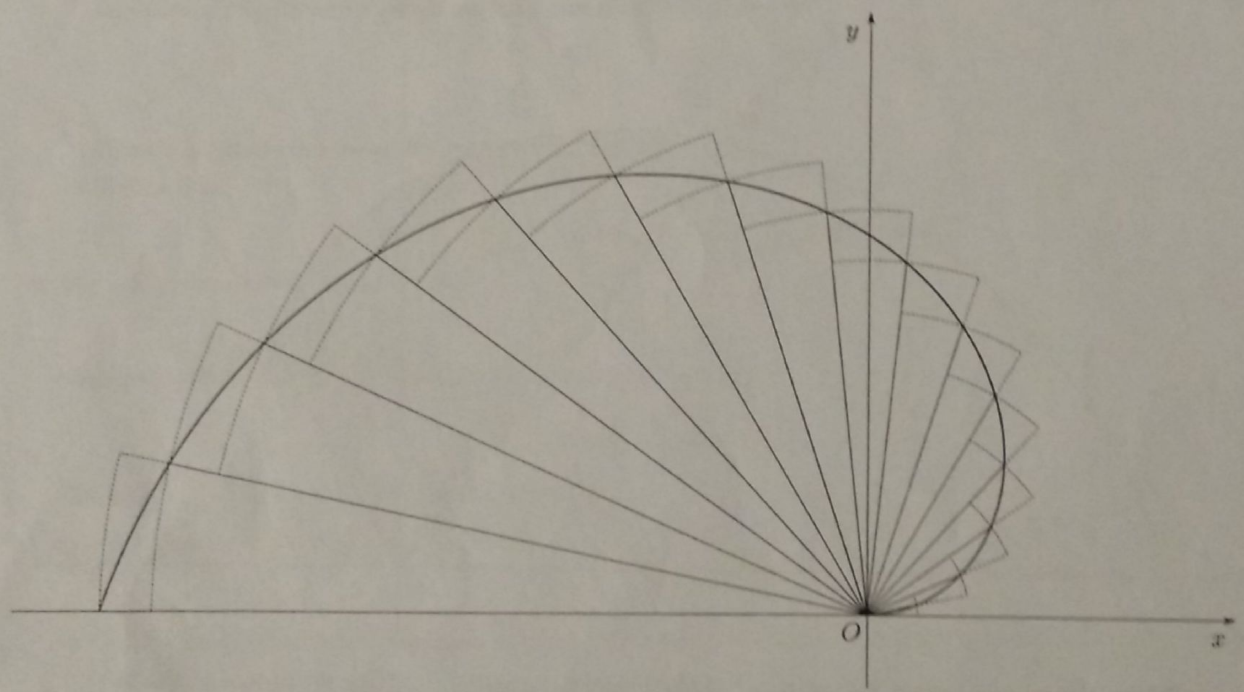

Figure 4 – Illustration of the inequality from Question 3b (shown here with $n = 15$)