SCAN 1 — Solution of Math Test #2

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Exercise 1.

1. Let $x \in \mathbb{R}$. From the addition formulas: $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$ and $\sinh(2x) = 2\sinh(x)\cosh(x)$ we obtain:

$$\tanh(2x) = \frac{\sinh(2x)}{\cosh(2x)} = \frac{2\sinh(x)\cosh(x)}{\cosh^2(x)+\sinh^2(x)} = \frac{2\dfrac{\sinh(x)}{\cosh(x)}}{1+\dfrac{\sinh^2(x)}{\cosh^2(x)}} = \frac{2\tanh(x)}{1+\tanh^2(x)}.$$

2. a) The domain of the function arctanh is $(-1, 1)$.

   b) Let $x \in \mathbb{R}$. In order to have this expression well-defined, we need:

$$\cos(x) \neq 0 \text{ and } \tan(x) \in (-1,1) \text{ and } \sin(2x) \in (-1,1).$$

   The first two conditions are equivalent to the existence of $k \in \mathbb{Z}$ such that $x \in (-\pi/4 + k\pi, \pi/4 + k\pi)$ and in this case, the last condition is fulfilled too since we'll have $2x \in (-\pi/2 + 2k\pi, \pi/2 + 2k\pi)$, and hence $\sin(x) \in (-1,1)$. Conclusion:

$$D = \bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{4} + k\pi, \frac{\pi}{4} + k\pi\right).$$

3. a) We show that π is a period of f: first notice that for all $x \in D$, $x + \pi \in D$. Now, for $x \in D$,

$$f(x+\pi) = 2\operatorname{arctanh}\big(\tan(x+\pi)\big) - \operatorname{arctanh}\big(\sin(2x+2\pi)\big) = 2\operatorname{arctanh}\big(\tan(x)\big) - \operatorname{arctanh}\big(\sin(2x)\big),$$

   since tan is periodic of period π and sin is periodic of period 2π.

   b) Let $x \in D$. From Question 1,

$$\begin{aligned}\tanh\Big(2\operatorname{arctanh}\big(\tan(x)\big)\Big) &= \frac{2\tanh\Big(\operatorname{arctanh}\big(\tan(x)\big)\Big)}{1+\tanh^2\Big(\operatorname{arctanh}\big(\tan(x)\big)\Big)}\\ &= \frac{2\tan(x)}{1+\tan^2(x)} = \frac{2\tan(x)\cos^2(x)}{\cos^2(x)+\sin^2(x)}\\ &= 2\sin(x)\cos(x) = \sin(2x).\end{aligned}$$

   c) From the previous question we conclude that for $x \in D$, $2\operatorname{arctanh}\big(\tan(x)\big) = \operatorname{arctanh}\big(\sin(2x)\big)$, hence $f(x) = 0$. Hence f is constant (equal to 0).

4. a) This expression is well-defined provided $x \in [1, +\infty)$ (for the domain of arccosh) and $\operatorname{arccosh}(x) \neq 0$, i.e., for $x \in E = (1, +\infty)$.

   b) The domain of arctanh is $(-1,1)$, and $(-1,1) \cap E = \emptyset$, hence there are no elements $x \in E$ for which $\operatorname{arctanh}(x)$ is defined; the equation has no solutions!

Exercise 2.

1. Let $n \in \mathbb{N}^*$. Then:

$$u_{n+1} - u_n = \frac{1}{(n+1)^2 + (n+1)\sin(2^{n+1})} = \frac{1}{(n+1)}\,\frac{1}{n+1+\sin(2^{n+1})}.$$

   Now, $1 + \sin(2^{n+1}) \geq 0$ and $n \geq 1$, $n + 1 + \sin(2^{n+1}) > 0$. Hence $u_{n+1} - u_n > 0$, hence the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

2. Let $k \in \mathbb{N}$ such that $n \geq 2$. If $k = 2$, the result is obvious. Otherwise, if $k \geq 3$,

$$\sum_{k=2}^{n} \frac{1}{k^2-k} = \sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) = \sum_{k=2}^{n}\frac{1}{k-1} - \sum_{k=2}^{n}\frac{1}{k} = \sum_{k=1}^{n-1}\frac{1}{k} - \sum_{k=2}^{n}\frac{1}{k} = 1 + \sum_{k=2}^{n-1}\frac{1}{k} - \sum_{k=2}^{n-1}\frac{1}{k} - \frac{1}{n} = 1 - \frac{1}{n}.$$

3. Observe that for all $k \geq 2$, since $\sin(2^k) \geq -1$ one has

$$k^2 + k\sin(2^k) \geq k^2 - k > 0,$$

   hence,

$$\frac{1}{k^2 + k\sin(2^k)} \leq \frac{1}{k^2-k}.$$

   Hence, for $n \in \mathbb{N}^*$ with $n \geq 2$,

$$u_n = \frac{1}{1+\sin(2)} + \sum_{k=2}^{n}\frac{1}{k^2+k\sin(2^k)} \leq \frac{1}{1+\sin(2)} + \sum_{k=2}^{n}\frac{1}{k^2-k} = \frac{1}{1+\sin(2)} + 1 - \frac{1}{n} \leq 1 + \frac{1}{1+\sin(2)}.$$

   We conclude that

$$\forall n \in \mathbb{N}^*,\ u_n \leq 1 + \frac{1}{1+\sin(2)},$$

   hence the sequence $(u_n)_{n\in\mathbb{N}^*}$ is bounded from above, an upper bound being

$$1 + \frac{1}{1+\sin(2)}.$$

   Since the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing, we conclude, by the Monotone Limit Theorem, that the sequence $(u_n)_{n\in\mathbb{N}^*}$ converges and that

$$\lim_{n\to+\infty} u_n \leq 1 + \frac{1}{1+\sin(2)}.$$

Exercise 3.

1. Let $\alpha \in \mathbb{R}$.

   a) By Euler's Formula and de Moivre's Formula,

$$\cos(3\alpha) + i\sin(3\alpha) = e^{3i\alpha} = \left(e^{i\alpha}\right)^3 = \big(\cos(\alpha) + i\sin(\alpha)\big)^3 = \cos^3(\alpha) + 3i\cos^2(\alpha)\sin(\alpha) - 3\cos(\alpha)\sin^2(\alpha) - i\sin^3(\alpha).$$

   Hence, taking the real and imaginary parts, we obtain:

$$\cos(3\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\sin^2(\alpha) \text{ and } \sin(3\alpha) = 3\cos^2(\alpha)\sin(\alpha) - \sin^3(\alpha).$$

   b) If $\cos(3\alpha) \neq 0$,

$$\tan(3\alpha) = \frac{\sin(3\alpha)}{\cos(3\alpha)} = \frac{3\cos^2(\alpha)\sin(\alpha) - \sin^3(\alpha)}{\cos^3(\alpha) - 3\cos(\alpha)\sin^2(\alpha)} = \frac{3\dfrac{\sin(\alpha)}{\cos(\alpha)} - \dfrac{\sin^3(\alpha)}{\cos^3(\alpha)}}{1 - 3\dfrac{\sin^2(\alpha)}{\cos^2(\alpha)}} = \frac{3\tan(\alpha) - \tan^3(\alpha)}{1 - 3\tan^2(\alpha)}.$$

2. a) $P(-1) = (-1)^3 - 3(-1)^2 - 3(-1) + 1 = -1 - 3 + 3 + 1 = 0$, hence -1 is a root of P.

   b) By a straightforward long division, we obtain:

$$\forall x \in \mathbb{R},\ P(x) = (x+1)(x^2 - 4x + 1).$$

   Now, for $x \in \mathbb{R}$,

$$x^2 - 4x + 1 = (x-2)^2 - 3 = \big(x - 2 - \sqrt{3}\big)\big(x - 2 + \sqrt{3}\big),$$

   hence the factored form of P in $\mathbb{R}$ is:

$$\forall x \in \mathbb{R},\ P(x) = (x+1)\big(x - 2 - \sqrt{3}\big)\big(x - 2 + \sqrt{3}\big),$$

   and the roots of P are: -1, $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

c) Notice that, from Question 1 one has:

$$1 = \tan(\pi/4) = \tan(3\pi/12) = \frac{3\tan(\pi/12) - \tan^3(\pi/12)}{1 - 3\tan^2(\pi/12)},$$

hence

$$1 - 3\tan^2(\pi/12) = 3\tan(\pi/12) - \tan^3(\pi/12)$$

hence

$$1 - 3\tan^2(\pi/12) - 3\tan(\pi/12) + \tan^3(\pi/12) = 0$$

hence $P(\tan(\pi/12)) = 0$. Hence $\tan(\pi/12)$ is a root of P.

d) Since tan is increasing on $[0, \pi)$ and since

$$0 < \pi/12 < \pi/4 < \pi,$$

we conclude that

$$0 < \tan(\pi/12) < 1.$$

Now, the only root of P that lies in $(0, 1)$ is $2 - \sqrt{3}$, we conclude that

$$\tan(\pi/12) = 2 - \sqrt{3}.$$

3. We know that for $x, y \in \mathbb{R}$ such that $\tan(x)$, $\tan(y)$ and $\tan(x - y)$ are defined we have:

$$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}.$$

Hence:

$$\tan(\pi/12) = \tan(\pi/3 - \pi/4) = \frac{\tan(\pi/3) - \tan(\pi/4)}{1 + \tan(\pi/3)\tan(\pi/4)}.$$

Now, $\tan(\pi/4) = 1$ and $\tan(\pi/3) = \sqrt{3}$, hence

$$\tan(\pi/12) = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = \frac{(\sqrt{3} - 1)^2}{(1 + \sqrt{3})(-1 + \sqrt{3})} = \frac{3 - 2\sqrt{3} + 1}{-1 + 3} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

Exercise 4.

1. a) Let $x \in A + B$. This means that $x = a + b$ for some $a \in A$ and $b \in B$. Since $a \in A$, we have $0 < a < 2$ and since $b \in B$, we have $1 \leq b \leq 2$. Hence

$$1 < a + b = x < 4,$$

hence $x \in (1, 4)$.

b) Let $x \in (1, 4)$.

- if $x < 3$: set $a = x - 1$ and $b = 1$. Since $1 < x < 3$, we have $0 < a < 2$, hence $a \in A$, and clearly $b \in B$. Since $x = a + b$ we conclude that $x \in A + B$.
- if $x \geq 3$: set $a = x - 3/2$ and $b = 3/2$. Since $3 \leq x < 4$, we have $3/2 \leq a < 5/2$, hence $a \in A$, and clearly $b \in B$. Since $x = a + b$ we conclude that $x \in A + B$.

c) The previous question shows that $(1, 4) \subset A + B$, and together with the first question we conclude that $A + B = (1, 4)$.

2. Since A and B are non-empty subsets, there exists $a \in A$ and $b \in B$. By definition, $a + b \in A + B$, hence $A + B \neq \emptyset$.

3. Since A and B are non-empty and bounded subsets (hence they are bounded from above), the LUB property of $\mathbb{R}$ guarantees that $\sup(A)$ and $\sup(B)$ exist in $\mathbb{R}$.

4. Let $x \in A + B$. By definition, there exists $a \in A$ and $b \in B$ such that $x = a + b$. Then,

$$x = a + b \leq \sup(A) + \sup(B).$$

Hence,

$$\forall x \in A + B,\ x \leq \sup(A) + \sup(B).$$

This shows that the set $A + B$ is bounded from above and that $\sup(A) + \sup(B)$ is an upper bound of the set $A + B$. By the LUB property of $\mathbb{R}$, we conclude that $\sup(A + B)$ exists in $\mathbb{R}$, and as $\sup(A + B)$ is the least upper bound of the set $A + B$, we must have

$$\sup(A + B) \leq \sup(A) + \sup(B).$$

a) Let $c \in \mathbb{R}$ such that $c < \sup(A)$, and assume that

$$\forall a \in A,\ c \geq a.$$

Then c is an upper bound of A. Since $\sup(A)$ is the least upper bound of A we must have $\sup(A) \leq c$, which is impossible, since $c < \sup(A)$. Hence the proposition

$$\forall a \in A,\ c \geq a$$

is false, hence

$$\exists a \in A,\ c < a.$$

b) Since we assumed that $\sup(A + B) < \sup(A) + \sup(B)$, we have $\sup(A + B) - \sup(B) < \sup(A)$. Using the previous question with $c = \sup(A + B) - \sup(B)$ we conclude that there exists $a \in A$ such that

$$\sup(A + B) - \sup(B) < a,$$

hence such that

$$\sup(A + B) < a + \sup(B),$$

c) Using the same reasoning as before: since $\sup(A + B) - a < \sup(B)$, there exists $b \in B$ such that

$$\sup(A + B) - a < b,$$

hence such that $\sup(A + B) < a + b$.

d) We showed that there exists $a \in A$ and $b \in B$ such that

$$\sup(A + B) < a + b,$$

but this is impossible since $a + b \in A + B$ and since $\sup(A + B)$ is an upper bound of the set $A + B$. We thus conclude that the proposition "$\sup(A + B) < \sup(A) + \sup(B)$" is false, hence we must have

$$\sup(A + B) \geq \sup(A) + \sup(B).$$

From the inequality of Question 4, we conclude that

$$\sup(A + B) = \sup(A) + \sup(B).$$