INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #2 — 1h30 *November 27, 2015*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1.

1. Show that
$$\forall x \in \mathbb{R},\ \tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)}.$$

2. a) Recall (without justifications) the domain of the function arctanh.

   b) Determine the maximal subset D of $\mathbb{R}$ such that for all $x \in D$ the expression
$$2\operatorname{arctanh}\big(\tan(x)\big) - \operatorname{arctanh}\big(\sin(2x)\big)$$
   is well-defined.

3. We define the function f as
$$\begin{aligned} f : D &\longrightarrow \mathbb{R} \\ x &\longmapsto 2\operatorname{arctanh}\big(\tan(x)\big) - \operatorname{arctanh}\big(\sin(2x)\big). \end{aligned}$$
   where D is the set you have determined in Question 2b.

   a) Show that f is a periodic function (and determine a period of f).

   b) Use Question 1 to show that for all $x \in D$ one has:
$$\tanh\Big(2\operatorname{arctanh}\big(\tan(x)\big)\Big) = \sin(2x).$$

   c) Deduce that f is constant.

4. a) Determine the maximal subset E of $\mathbb{R}$ such that for all $x \in E$ the expression
$$\frac{\operatorname{arcsinh}(x)}{\operatorname{arccosh}(x)}$$
   is well-defined. Justify with care.

   b) Find all the elements $x \in E$ such that
$$\operatorname{arctanh}(x) = \frac{\operatorname{arcsinh}(x)}{\operatorname{arccosh}(x)}.$$

Exercise 2. Let $(u_n)_{n\in\mathbb{N}^*}$ be the sequence defined by
$$\forall n \in \mathbb{N}^*,\ u_n = \sum_{k=1}^{n} \frac{1}{k^2 + k\sin\big(2^k\big)}.$$

1. Show that the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

2. Show that for all $n \in \mathbb{N}$ such that $n \geq 2$ one has
$$\sum_{k=2}^{n} \frac{1}{k^2 - k} = 1 - \frac{1}{n}.$$
   Hint: use the decomposition
$$\frac{1}{k^2 - k} = \frac{1}{k-1} - \frac{1}{k}.$$

3. Deduce that the sequence $(u_n)_{n\in\mathbb{N}^*}$ is bounded from above (and determine explicitly an upper bound). What can you conclude about the sequence $(u_n)_{n\in\mathbb{N}^*}$? justify your answer.

Exercise 3. The goal of this exercise is to determine the value of $\tan(\pi/12)$.

1. Let $a \in \mathbb{R}$.

   a) Express $\cos(3a)$ and $\sin(3a)$ in terms of $\cos(a)$ and $\sin(a)$.

   b) We assume moreover that $\cos(3a) \neq 0$ (and hence $\cos(a) \neq 0$). Show that

$$\tan(3a) = \frac{3\tan(a) - \tan^3(a)}{1 - 3\tan^2(a)}.$$

2. Let P be the polynomial function defined by

$$\begin{array}{rccl} P: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^3 - 3x^2 - 3x + 1. \end{array}$$

   a) Show that -1 is a root of P.

   b) Deduce the factored form (in $\mathbb{R}$) of P, and deduce the roots of P.

   c) Show, using Question 1b, that $\tan(\pi/12)$ is a root of P.

   d) Deduce the value of $\tan(\pi/12)$.

3. A much quicker way: use the subtraction formula for tan and the fact that $\pi/12 = \pi/3 - \pi/4$ to obtain the value of $\tan(\pi/12)$ in a more direct way.

Exercise 4. Let A and B be two *non-empty* and *bounded* subsets of $\mathbb{R}$. We define the set subset $A + B$ of $\mathbb{R}$ as:

$$A + B = \{a + b;\ a \in A \text{ and } b \in B\}.$$

1. In this question only we determine the set $A + B$ explicitly in the special case $A = (0,2)$ and $B = [1,2]$.

   a) Show that $A + B \subset (1,4)$.

   b) Show that:

$$\forall x \in (1,4),\ \exists a \in A,\ \exists b \in B,\ a + b = x.$$

$\Rightarrow A + B = (1,4)$

   Hint: you might want to separate to cases: whether $x < 3$, or $x \geq 3$.

   c) Conclude.

We're now back to the general case, where A and B are just non-empty subsets of $\mathbb{R}$, not necessarily the ones considered in Question 1.

2. Show that $A + B \neq \emptyset$.

3. Show that $\sup(A)$ and $\sup(B)$ exist (in $\mathbb{R}$).

4. Prove that $\sup(A + B)$ exists (in $\mathbb{R}$) and that

$$\sup(A + B) \leq \sup(A) + \sup(B).$$

5. The goal of this question is to prove that $\sup(A + B) = \sup(A) + \sup(B)$. We proceed by contradiction, and hence we assume that $\sup(A + B) < \sup(A) + \sup(B)$.

   a) Let $c \in \mathbb{R}$ such that $c < \sup(A)$. Show by contradiction that there exists $a \in A$ such that $c < a$.

   b) Deduce that there exists $a \in A$ such that $\sup(A + B) < a + \sup(B)$.

   c) Show that there exists $b \in B$ such that $\sup(A + B) < a + b$.

   d) Conclude.