INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #4 — 2h

March 25, 2016

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The goal of this exercise is to compute the value of $\cos(1/3)$ correct to 5 decimal places.

1. Show that

$$-\frac{1}{3^6 6!} < \cos(1/3) - \left(1 - \frac{1}{3^2 2} + \frac{1}{3^4 24}\right) < 0.$$

2. You're given:

$$1 - \frac{1}{3^2 2} + \frac{1}{3^4 24} = \frac{1837}{1944} = 0.944\overline{958847736625514403292181069}, \qquad \frac{1}{3^6 6!} = \frac{1}{524880} < 2 \cdot 10^{-6},$$

(where the bar over the digits means that these digits are infinitely repeated). Deduce a numerical lower bound and upper bound of $\cos(1/3)$ with 6 significant figures, and deduce the value of $\cos(1/3)$ correct to 5 decimal places.

Exercise 2.

1. Find the simplest equivalent of $\ln(1 + x) - \ln(x)$ as $x \to +\infty$.
2. Find the simplest equivalent of $x^2\left(e^{1/(x+1)} - e^{1/x}\right)$ as $x \to +\infty$.
3. Find the simplest equivalent of $\ln(e^x - x)$ as $x \to 0$.

Exercise 3 (*Kolmogorov's Inequality*). Let $a \in \mathbb{R}$ and let $f : (a, +\infty) \longrightarrow \mathbb{R}$ be a function such that:

- f is twice differentiable,
- f is bounded; we define: $M_0 = \sup|f|$,
- f'' is bounded; we define: $M_2 = \sup|f''|$.

The goal of this exercise is to show that $|f'|$ is bounded, and that

$$(\bullet) \qquad |f'| \leq 2\sqrt{M_0 M_2}$$

(this inequality is known as *Kolmogorov's inequality*).

1. Let $x \in (a, +\infty)$ and $u \in \mathbb{R}_+^*$. Apply Taylor–Lagrange formula on $[x, x + u]$ to show that

$$\left|f'(x)\right| \leq \frac{2M_0}{u} + \frac{u}{2}M_2.$$

2. Deduce that if $M_2 = 0$, then Inequality $(\bullet)$ is fulfilled.
3. In this question we suppose that $M_2 \neq 0$. We define the function

$$g : \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \dfrac{2M_0}{u} + \dfrac{u}{2}M_2. \end{array}$$

Solve the equation $g'(u) = 0$ and deduce that

$$\min g = 2\sqrt{M_0 M_2}.$$

Conclude.

$\ln(1+x) = x + o(x)$

Exercise 4. Let $\lambda \in \mathbb{R}$. We define the function f as

$$f : \begin{array}{ccc} (-\pi, 0) \cup (0, \pi) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{\sin x} - \dfrac{e^{\lambda x}}{x}. \end{array}$$

1. Show that

$$f(x) \underset{x \to 0}{=} -\lambda + \frac{1 - 3\lambda^2}{6} x - \frac{\lambda^3}{6} x^2 + \frac{-15\lambda^4 + 7}{360} x^3 + o\left(x^3\right).$$

2. Deduce that f possesses an extension by continuity at 0, denoted by $\tilde{f}$, and give the value of $\tilde{f}(0)$.

3. Explain why $\tilde{f}$ is differentiable at 0, and give the value of $\tilde{f}'(0)$. Explicit an equation of the tangent line Δ to the graph of $\tilde{f}$ at $\left(0, \tilde{f}(0)\right)$.

4. Give (in terms of λ) the relative position of the graph of $\tilde{f}$ with respect to Δ, in a neighborhood of 0.

Exercise 5.

1. Let $a, b \in \mathbb{R}_+^*$ with $a < b$. Use the Mean Value Theorem for Integrals (MVTI) to show that

$$(b - a)\ln(a) \leq \int_a^b \ln(x)\, dx \leq (b - a)\ln(b),$$

   and deduce that for all $n \in \mathbb{N}$ with $n \geq 2$,

$$\int_{n-1}^{n} \ln(x)\, dx \leq \ln(n) \leq \int_n^{n+1} \ln(x)\, dx.$$

2. Show that for all $N \in \mathbb{N}$ with $N \geq 2$,

$$\int_1^N \ln(x)\, dx \leq \ln(N!) \leq \int_2^{N+1} \ln(x)\, dx.$$

3. Give an antiderivative of ln on $\mathbb{R}_+^*$ (no justifications required).

4. Deduce that for all $N \in \mathbb{N}$ with $N \geq 2$,

$$N^N e^{1-N} \leq N! \leq \frac{1}{4}(N+1)^{N+1} e^{1-N}.$$

   Is this inequality still valid for $N = 1$? for $N = 0$?

Exercise 6. The purpose of this exercise is to study the Taylor expansion of the solutions of class C^∞ on $\mathbb{R}_+$ of the following differential equation

$$(*) \qquad x^2 y'(x) + y(x) = x^2.$$

Let $y : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a solution of class C^∞ of Equation $(*)$. For $n \in \mathbb{N}$ we define

$$u_n = y^{(n)}(0).$$

1. Deduce from Equation $(*)$ that $u_0 = 0$.

2. By differentiating Equation $(*)$, determine the value of u_1 and of u_2, and give the second order Taylor–Young expansion of y at 0^+.

3. a) Let $n \in \mathbb{N}$ with $n \geq 3$. Show that

$$\forall x \in \mathbb{R}_+, \; x^2 y^{(n+1)}(x) + (1 + 2nx) y^{(n)}(x) + n(n-1) y^{(n-1)}(x) = 0.$$

   b) Deduce, for $n \geq 3$, a relation between u_n and u_{n-1}.

   c) Deduce, for $n \geq 3$, an explicit expression of u_n.

4. For $n \geq 3$, give the n-th order Taylor–Young expansion of y at 0^+.

$\frac{1}{1-x}$

$1 + n - 2 = 3 + h -$