INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 – Math Test #5 – 2h

May 20, 2016

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The questions of this exercise are independent from each other.

1. Compute the value of the following integral:

~~sint~~

cost - cost + t sin ~~x~~

$$I = \int_0^{\pi} t\sin(t)\,\mathrm{d}t.$$

2. Determine the value of the following indefinite integral

$$\int \frac{\mathrm{d}x}{x^2 - 3x + 2}$$

and specify the domain of validity.

3. Use the substitution $t = \mathrm{e}^x \cos x$ to compute the value of the following integral:

$$J = \int_0^{\pi/4} \left(1 + \mathrm{e}^x \cos x\right)(1 - \tan x)\,\mathrm{d}x.$$

Exercise 2. We define the sequences $(I_n)_{n\in\mathbb{N}}$ and $(J_n)_{n\in\mathbb{N}}$ by

$$\forall n \in \mathbb{N},\ I_n = \int_0^{\pi/2} \cos(nt)\big(\cos(t)\big)^n\,\mathrm{d}t \quad \text{and} \quad J_n = \int_0^{\pi/2} \cos\big((n+2)t\big)\big(\cos(t)\big)^n\,\mathrm{d}t.$$

1. Compute the value of I_0, J_0 and I_1.
2. a) Show that

$$\forall n \in \mathbb{N},\ I_n - J_n = 2\int_0^{\pi/2} \sin\big((n+1)t\big)\sin(t)\big(\cos(t)\big)^n\,\mathrm{d}t.$$

   b) Deduce that

$$\forall n \in \mathbb{N},\ I_n - J_n = 2I_{n+1}.$$

3. Show that

$$\forall n \in \mathbb{N},\ I_n + J_n = 2I_{n+1}.$$

4. Deduce, for $n \in \mathbb{N}$, an explicit expression of I_n and J_n.

Exercise 3. Use the Mean Value Theorem for Integrals (MVT2) to determine the value of the following limit:

$$\lim_{x\to 0^+} \int_x^{2x} \frac{\sin t}{t^2}\,\mathrm{d}t.$$

Exercise 4. Let $m \in \mathbb{R}$. Solve the following linear system:

$$(S)\quad \begin{cases} (2-m)x - y - z = 0 \\ -x + (2-m)y - z = 0 \\ -x - y + (2-m)z = 0 \end{cases}$$

You will also mention the rank of the system (depending on the value of m).

Exercise 5. Let $E = \mathbb{R}^4$, define the vectors

$$u = (1, 1, -1, -1), \qquad v = (1, -3, 4, 1), \qquad w = (2, -2, 3, 0).$$

and the sets

$$F = \left\{(x, y, z, t) \in \mathbb{R}^4 \,\middle|\, 2x + z + t = 0,\ x + y + z + t = 0\right\}$$
$$G = \mathrm{Span}\{u, v, w\}.$$

1. Show that F is a linear subspace of E. Is the set G a linear subspace of E?
2. By solving the linear system
$$\begin{cases} 2x \qquad + z + t = 0 \\ x + y + z + t = 0, \end{cases}$$
determine a basis $\mathscr{A}$ of F, and deduce the dimension of F.
3. Determine a basis $\mathscr{C}$ of G and the dimension of G.
4. Determine $F \cap G$ explicitly. Are F and G independent subspaces of E?
5. What is the dimension of $F + G$? Determine a basis of $F + G$.

Exercise 6. We define the family $\mathscr{B}$ of vectors of $\mathbb{R}^3$ as

$$\mathscr{B} = \big((2, 1, 1), (1, 2, 1), (1, 1, 2)\big)$$

and the family $\mathscr{C}$ of vectors of $\mathbb{R}^2$ as

$$\mathscr{C} = \big((2, 1), (1, 2)\big).$$

1. Show that $\mathscr{B}$ is a basis of $\mathbb{R}^3$ and that $\mathscr{C}$ is a basis of $\mathbb{R}^2$.
2. For $u = (x, y, z) \in \mathbb{R}^3$, determine the coordinates of u in $\mathscr{B}$, that is, $[u]_{\mathscr{B}}$.
3. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear mapping such that the matrix of f in the basis $\mathscr{B}$ and $\mathscr{C}$ is
$$[f]_{\mathscr{B},\mathscr{C}} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}.$$
For $u = (x, y, z) \in \mathbb{R}^3$, determine an explicit expression of $f(u)$.

Exercise 7. Let E and F be two vector spaces over the commutative field $\mathbb{K}$. Let $f : E \longrightarrow F$ be a linear mapping. Show that:

$$f \text{ is injective} \iff \mathrm{Ker}\, f = \{0_E\}.$$