SCAN 1 — Solution of Final Exam

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Exercise 1. Let $N \in \mathbb{N}$. Then:

$$S_N = \sum_{n=0}^{N} (2^n + n) = \sum_{n=0}^{N} 2^n + \sum_{n=0}^{N} n.$$

Using the sum of the terms of a geometric sequence we obtain

$$\sum_{n=0}^{N} 2^n = \frac{1 - 2^{N+1}}{1-2} = 2^{N+1} - 1,$$

and using the sum of the consecutive integers we obtain

$$\sum_{n=0}^{N} n = \frac{N(N+1)}{2}.$$

Hence,

$$S_N = 2^{N+1} - 1 + \frac{N(N+1)}{2}.$$

Now, we also have

$$S_N = \sum_{n=0}^{N} (a_{n+1} - a_n) = a_{N+1} - a_0 = a_{N+1} \qquad \textit{(since } a_0 = 0\textit{)},$$

hence

$$a_{N+1} = 2^{N+1} - 1 + \frac{N(N+1)}{2},$$

hence,

$$\forall n \in \mathbb{N}^*,\ a_n = 2^n - 1 + \frac{(n-1)n}{2}.$$

It turns out that this formula is also correct for $n = 0$, hence

$$\forall n \in \mathbb{N},\ a_n = 2^n - 1 + \frac{(n-1)n}{2}.$$

Exercise 2. We notice that 4 is an obvious eigenvalue since

$$\operatorname{rk}(A - 4I_3) = \operatorname{rk}\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} = 1 \neq 3.$$

We also conclude that the dimension of the eigenspace E_4 is $\dim E_4 = \dim \operatorname{Ker}(A - 4I_3) = 3 - 1 = 2$. Hence the multiplicity $\operatorname{mult}(4)$ of the eigenvalue 4 is at least 2. We're missing only one eigenvalue, say λ so we use the trace of A: $\operatorname{tr}(A) = 10 = 4 + 4 + \lambda$, hence $\lambda = 10 - 2 \times 4 = 2 \neq 4$. Now, we know that the multiplicity of 2 is at least 1, and the sum of the multiplicities is 3, so we must have $\operatorname{mult}(2) = 1$ and $\operatorname{mult}(4) = 2$. Since $\operatorname{mult}(4) = 2 = \dim E_4$ (and we don't check anything for eigenvalues of multiplicity 1, the dimension of the eigenspace being automatically 1), we conclude that the matrix A is diagonalizable.
We now determine a basis of eigenvectors:

- E_4: let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then:

$$AX = 4X \iff -x + y - z = 0 \iff \begin{cases} x = y - z \\ y = y \\ z = z \end{cases} \iff X = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

- E_4: let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then:

$$AX = 2X \iff \begin{cases} x + y - z = 0 \\ 2y = 0 \\ -x + y + z = 0 \end{cases} \iff \begin{cases} x = z \\ y = 0 \\ z = z \end{cases} \iff X = z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, if we set

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

we have $A = PDP^{-1}$.
If we set, for $t \in \mathbb{R}$, $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, then the linear system (S) can be written as

$$X'(t) = AX(t),$$

i.e.,

$$X'(t) = PDP^{-1}X(t).$$

We set $U(t) = P^{-1}X(t)$, and we give a name to the components of U, say $U(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}$. Then, since P is a constant matrix, the linear system (S) is equivalent to

$$U'(t) = DU(t),$$

namely

$$\text{S}' \begin{cases} u'(t) = 4u(t) \\ v'(t) = 4v(t) \\ w'(t) = 2w(t) \end{cases} \iff \begin{cases} u(t) = C_1 e^{4t} \\ v(t) = C_2 e^{4t} \\ w(t) = C_3 e^{2t}, \end{cases} \qquad \text{for some } C_1, C_2, C_3 \in \mathbb{R}.$$

Since $X(t) = PU(t)$, we conclude that

$$\begin{cases} x(t) = (C_1 - C_2)e^{4t} + C_3 e^{2t} \\ y(t) = C_1 e^{4t} \\ z(t) = C_2 e^{4t} + C_3 e^{2t}. \end{cases}$$

Exercise 3.

Part I

1. - Let $x \in [1, +\infty)$. Notice that

$$\forall t \in [1, x],\ \frac{\ln(t)}{1+t^2} \geq 0$$

(since $x \geq 1$) hence, integrating from 1 to x (with $1 \leq x$) yields

$$F(x) = \int_1^x \frac{\ln(t)}{1+t^2}\, dt \geq 0.$$

- Let $x \in (0, 1]$. Notice that

$$\forall t \in [x, 1],\ \frac{\ln(t)}{1+t^2} \leq 0$$

(since $x \leq 1$) hence, integrating from 1 to x (with $1 \geq x$, remember that will change the sign of the inequality) yields

$$F(x) = \int_1^x \frac{\ln(t)}{1+t^2}\, dt \geq 0.$$

Conclusion:

$$\forall x \in \mathbb{R}_+^*,\ F(x) \geq 0.$$

2. Since the function $h : t \longmapsto \dfrac{\ln(t)}{1+t^2}$ is continuous on $\mathbb{R}_+^*$, h possesses an antiderivative on $\mathbb{R}_+^*$, say H: this means that H is a differentiable function on $\mathbb{R}_+^*$ and that

$$\forall x \in \mathbb{R}_+^*,\ H'(x) = h(x).$$

Now, the function H being an antiderivative of h, by the Fundamental Theorem of Calculus, one has

$$\forall x \in \mathbb{R}_+^*,\ F(x) = H(x) - H(1).$$

Since H is differentiable on $\mathbb{R}_+^*$, we conclude that F is differentiable on $\mathbb{R}_+^*$ and that

$$\forall x \in \mathbb{R}_+^*,\ F'(x) = H'(x) = \frac{\ln(x)}{1+x^2},$$

from which we observe that F' is of class C^∞, hence F is of class C^∞ (and in particular of class C^1).

3. Let $x \in \mathbb{R}_+^* \setminus \{1\}$. We use the subsitution $u = 1/t$: then $\mathrm{d}t = -\mathrm{d}u/u^2$ and

$$F(x) = \int_1^x \frac{\ln(t)}{1+t^2}\,\mathrm{d}t = \int_1^{1/x} \frac{\ln(1/u)}{1+(1/u)^2}\left(-\frac{\mathrm{d}u}{u^2}\right) = \int_1^{1/x} \frac{\ln(u)}{1+u^2}\,\mathrm{d}u = F(1/x).$$

Of course, this equality is obviously true if $x = 1$.

4. a) Clearly,

$$\lim_{t\to 0} \varphi(t) = \lim_{t\to 0} \frac{\arctan(t)}{t} = \lim_{t\to 0} \frac{\arctan(t) - \arctan(0)}{t - 0} = \arctan'(0) = 1,$$

hence φ possesses an extension by continuity at 0, namely the function $\tilde{\varphi}$ defined by

$$\begin{aligned} \tilde{\varphi} : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} \dfrac{\arctan(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases} \end{aligned}$$

b) Since $\tilde{\varphi}$ is continuous on $\mathbb{R}$, we know that the function Φ is an antiderivative of $\tilde{\varphi}$ on $\mathbb{R}$ (it's actually *the* antiderivative of $\tilde{\varphi}$ that vanishes at 0), and in particular Φ is differentiable hence continuous on $\mathbb{R}$.

c) Let $x \in \mathbb{R}_+^*$. By an integration by parts (differentiating $t \longmapsto \ln(t)$):

$$\begin{aligned} F(x) &= \int_1^x \frac{\ln(t)}{1+t^2}\,\mathrm{d}t = \big[\ln(t)\arctan(t)\big]_{t=1}^{t=x} - \int_1^x \frac{\arctan(t)}{t}\,\mathrm{d}t \\ &= \ln(x)\arctan(x) - \ln(1)\arctan(1) - \int_1^0 \tilde{\varphi}(t)\,\mathrm{d}t - \int_1^x \tilde{\varphi}(t)\,\mathrm{d}t \\ &= \ln(x)\arctan(x) + \Phi(1) - \Phi(x). \end{aligned}$$

d) Since

$$\arctan(x)\ln(x) \underset{x\to 0^+}{\sim} x\ln(x) \underset{x\to 0^+}{\longrightarrow} 0,$$

and since Φ is continuous, we conclude that

$$\lim_{x\to 0^+} F(x) = 0 - \Phi(0) - \Phi(1) = -\Phi(1) \qquad (\textit{since } \Phi(0) = 0).$$

Hence F possesses an extension by continuity on $\mathbb{R}_+$, namely the function $\tilde{F}$ defined by

$$\begin{aligned} \tilde{F} : \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} F(x) & \text{if } x \neq 0 \\ -\Phi(1) & \text{if } x = 0. \end{cases} \end{aligned}$$

5. For all $x \in \mathbb{R}_+^*$,

$$\frac{\tilde{F}(x) - \tilde{F}(0)}{x} = \frac{\arctan(x)\ln(x)}{x} - \frac{\Phi(x) - \Phi(0)}{x}.$$

Now, the limit

$$\lim_{x\to 0^+} \frac{\Phi(x) - \Phi(0)}{x} = \Phi'(0) = \tilde{\varphi}(0) = 1$$

exists in $\mathbb{R}$, since Φ is differentiable, but

$$\frac{\arctan(x)\ln(x)}{x} \underset{x\to 0^+}{\sim} \ln(x) \underset{x\to 0^+}{\longrightarrow} -\infty,$$

hence

$$\lim_{x\to 0^+} \frac{\tilde{F}(x) - \tilde{F}(0)}{x - 0} = -\infty,$$

hence $\tilde{F}$ is not differentiable (from the right) at 0.

6. Remark: we can't use the Mean Value Theorem directly on $[0,1]$, since the function ln is not continuous on $[0,1]$ (as ln is not even defined at 0). Let $x \in (0,1]$. The function ln is continuous on $[x,1]$ and the function $t \longmapsto 1/(1+t^2)$ is (piecewise) continuous and positive on $[x,1]$ hence, by the Mean Value Theorem, there exists $c_x \in [x,1]$ such that

$$F(x) = \int_1^x \frac{\ln(t)}{1+t^2}\,\mathrm{d}t = \frac{1}{1+c_x^2}\int_1^x \ln(t)\,\mathrm{d}t = \frac{1}{1+c_x^2}\big[t\ln(t) - t\big]_{t=1}^{t=x} = \frac{1}{1+c_x^2}\big(x\ln(x) - x + 1\big).$$

Notice that since $c_x \in [x,1] \subset (0,1]$, one has

$$\frac{1}{2} \leq \frac{1}{1+c_x^2} \leq 1.$$

Moreover, since $\lim_{x\to 0} x\ln(x) - x + 1 = 1$, there exists $x_0 \in (0,1]$ such that

$$\forall x \in (0,x_0],\ x\ln(x) - x + 1 \geq 0,$$

so that

$$\forall x \in (0,x_0],\ \frac{1}{2}\big(x\ln(x) - x + 1\big) \leq \frac{1}{1+c_x^2}\big(x\ln(x) - x + 1\big) \leq x\ln(x) - x + 1.$$

Hence,

$$\forall x \in (0,x_0],\ \frac{1}{2}\big(x\ln(x) - x + 1\big) \leq F(x) \leq x\ln(x) - x + 1$$

and taking the limit as $x \to 0^+$ yields

$$\frac{1}{2} \leq \tilde{F}(0) \leq 1$$

Part II

1. Let $k \in \mathbb{N}$ and $x \in \mathbb{R}_+^*$. By an integration by parts, differentiating the ln,

$$\begin{aligned} I_k(x) &= \int_1^x t^k \ln(t)\,\mathrm{d}t \\ &= \left[\frac{t^{k+1}}{k+1}\ln(t)\right]_{t=1}^{t=x} - \int_1^x \frac{t^{k+1}}{k+1}\frac{1}{t}\,\mathrm{d}t \\ &= \frac{x^{k+1}}{k+1} - \frac{1}{k+1}\int_1^x t^k\,\mathrm{d}t \\ &= \frac{x^{k+1}}{k+1} - \frac{1}{k+1}\left[\frac{t^{k+1}}{k+1}\right]_{t=1}^{t=x} \\ &= \frac{x^{k+1}}{k+1} - \frac{1}{(k+1)^2}\big(x^{k+1} - 1\big). \end{aligned}$$

Hence,

$$\lim_{x\to 0^+} I_k(x) = \frac{1}{(k+1)^2}$$

2. Let $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then:

$$\sum_{k=0}^{n}(-1)^k t^{2k} = \sum_{k=0}^{n}(-t^2)^k = \frac{1-(-t^2)^{n+1}}{1+t^2} = \frac{1-(-1)^{n+1}t^{2n+2}}{1+t^2}.$$

Hence,

$$\frac{1}{1+t^2} - \sum_{k=0}^{n}(-1)^k t^{2k} = (-1)^{n+1}\frac{t^{2n+2}}{1+t^2},$$

and taking the absolute value yields

$$\left|\frac{1}{1+t^2} - \sum_{k=0}^{n}(-1)^k t^{2k}\right| = \frac{t^{2n+2}}{1+t^2} \le t^{2n+2},$$

since $1+t^2 > 0$.

3. Let $n \in \mathbb{N}$ and $t \in (0,1]$. Multiplying the previous inequality by $|\ln(t)| = -\ln(t) \ge 0$ yields

$$\left|\frac{\ln(t)}{1+t^2} - \sum_{k=0}^{n}(-1)^k t^{2k}\ln(t)\right| \le -\ln(t)t^{2n+2},$$

Now, for $x \in (0,1]$,

$$\begin{aligned}
\left|\int_1^x \frac{\ln(t)}{1+t^2}\,\mathrm{d}t - \sum_{k=0}^{n}(-1)^k I_{2k}(x)\right| &= \left|\int_1^x \frac{\ln(t)}{1+t^2}\,\mathrm{d}t - \sum_{k=0}^{n}(-1)^k \int_1^x t^{2k}\ln(t)\,\mathrm{d}t\right| \\
&= \left|\int_1^x \left(\frac{\ln(t)}{1+t^2} - \sum_{k=0}^{n}(-1)^k t^{2k}\ln(t)\right)\mathrm{d}t\right| \\
&= \left|\int_x^1 \left(\frac{\ln(t)}{1+t^2} - \sum_{k=0}^{n}(-1)^k t^{2k}\ln(t)\right)\mathrm{d}t\right| \\
&\le \int_x^1 \left|\frac{\ln(t)}{1+t^2} - \sum_{k=0}^{n}(-1)^k t^{2k}\ln(t)\right|\mathrm{d}t \quad \text{by the Triangle Inequality, since } x<1 \\
&\le \int_x^1 -\ln(t)t^{2n+2}\,\mathrm{d}t \quad \text{by the previous inequality, since } x<1 \\
&= \int_1^x \ln(t)t^{2n+2}\,\mathrm{d}t = I_{2n+2}(x).
\end{aligned}$$

Hence, taking the limit as $x \to 0^+$ yields

$$\left|\tilde{F}(0) - \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)^2}\right| \le \frac{1}{(2n+3)^2}.$$

4. In the special case $n = 500$, we obtain

$$\left|\tilde{F}(0) - \sum_{k=0}^{500}\frac{(-1)^k}{(2k+1)^2}\right| \le \frac{1}{1003^2} < 10^{-6},$$

i.e.,

$$\sum_{k=0}^{500}\frac{(-1)^k}{(2k+1)^2} - 10^{-6} < \tilde{F}(0) < \sum_{k=0}^{500}\frac{(-1)^k}{(2k+1)^2} + 10^{-6}.$$

Now, from the result given by Maple, we conclude that

$$0.915966 < \sum_{k=0}^{500}\frac{(-1)^k}{(2k+1)^2} < 0.915967,$$

hence,

$$0.915965 < \tilde{F}(0) < 0.915968.$$

We hence obtain the approximation of $\tilde{F}(0)$ correct to 5 decimal places:[1]

$$\tilde{F}(0) = 0.91596\ldots$$

Exercise 4.

Part I

1. Since A is a triangular matrix, we directly read the eigenvalues of A and their multiplicities on the diagonal: A possesses a unique eigenvalue, namely 2, of multiplicity 3. The matrix A is not diagonalizable for if it were, A would be $2I_3$ (as A possesses a unique eigenvalue), which is false.

2.

$$N^2 = \begin{pmatrix}0&0&2\\0&0&0\\0&0&0\end{pmatrix}, \qquad N^3 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Hence,

$$\forall n \ge 3,\ N^n = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

3. Let $n \ge 2$. Since the matrices N and $2I_3$ commute (i.e., $N(2I_3) = 2M = (2I_3)N$), the Binomial Theorem states that

$$A^n = (N+2I_3)^n = \sum_{k=0}^{n}\binom{n}{k}N^k(2I_3)^{n-k}.$$

Now, since

$$\forall k \ge 3,\ N^k = 0_{\mathcal{M}_3(\mathbb{R})},$$

we conclude that

$$\begin{aligned}
A^n = (N+2I_3)^n &= \sum_{k=0}^{2}\binom{n}{k}N^k(2I_3)^{n-k} \\
&= \binom{n}{0}N^0(2I_3)^n + \binom{n}{1}N^1(2I_3)^{n-1} + \binom{n}{2}N^2(2I_3)^{n-2} \\
&= 2^n I_3 + 2^{n-1}nN + \frac{n(n-1)}{2}2^{n-2}N^2 \\
&= 2^n I_3 + 2^{n-1}nN + 2^{n-3}n(n-1)N^2 \\
&= \begin{pmatrix}2^n & 2^{n-1}n & 2^{n-2}n(n-1)\\0 & 2^n & 2^n n\\0&0&2^n\end{pmatrix}.
\end{aligned}$$

In the cases $n=1$ and $n=0$, we read

$$A^1 = \begin{pmatrix}2&1&0\\0&2&2\\0&0&2\end{pmatrix}, \qquad A^0 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

which is true.

Part II

1. By definition of Span, the family $\mathscr{B}$ is a generating family of F. We hence only need to show that the family $\mathscr{B}$ is an independent family. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha f_0 + \alpha f_1 + \alpha f_2 = 0_E$, i.e., such that

$$\forall x \in \mathbb{R},\ (\alpha + \beta x + \gamma x^2)e^{2x} = 0.$$

[1] The Maple command `evalf[100](Catalan)` yields

0.9159655941772190150546035149323841107741493742816721342664981196217630197762547694793565129261151062.

Since the exponential never vanishes, we then have

$$\forall x \in \mathbf{R},\ \alpha + \beta x + \gamma x^2 = 0,$$

and since the function $x \longmapsto \alpha + \beta x + \gamma x^2$ is a polynomial function, we conclude that all its coefficients must be nil, namely that $\alpha = \beta = \gamma = 0$. Hence $\mathscr{B}$ is an independent family, hence $\mathscr{B}$ is a basis of F. We then conclude that $\dim F = \#\mathscr{B} = 3$.

2. Let $f \in F$, say $f = \alpha f_0 + \beta f_1 + \gamma f_2$, i.e.,

$$\forall x \in \mathbf{R},\ f(x) = \left(\alpha + \beta x + \gamma x^2\right)\mathrm{e}^{2x}.$$

Then

$$\forall x \in \mathbf{R},\ f'(x) = \left(2\alpha + \beta + (2\beta + 2\gamma)x + 2\gamma x^2\right)\mathrm{e}^{2x},$$

and we conclude that

$$f' = (2\alpha + \beta)f_0 + (2\beta + 2\gamma)f_1 + 2\gamma f_2 \in \mathrm{Span}\{f_0, f_1, f_2\} = F.$$

Hence ψ is well-defined.

We now show that ψ is linear: let $f, g \in F$ and $\lambda \in \mathbf{R}$. Then

$$\psi(f + \lambda g) = (f + \lambda g)' = f' + \lambda g' = \psi(f) + \lambda\psi(g).$$

3. From the previous computation, i.e.,

$$\psi(\alpha f_0 + \beta f_1 + \gamma f_2) = (2\alpha + \beta)f_0 + (2\beta + 2\gamma)f_1 + 2\gamma f_2,$$

we conclude that

$$\left[\psi(f_0)\right]_{\mathscr{B}} = \begin{pmatrix}2\\0\\0\end{pmatrix}, \qquad \left[\psi(f_1)\right]_{\mathscr{B}} = \begin{pmatrix}1\\2\\0\end{pmatrix}, \qquad \left[\psi(f_2)\right]_{\mathscr{B}} = \begin{pmatrix}0\\2\\2\end{pmatrix},$$

hence

$$[\psi_2]_{\mathscr{B}_2} = \begin{pmatrix}2 & 1 & 0\\0 & 2 & 2\\0 & 0 & 2\end{pmatrix} = A.$$

4. We notice that $f \in F$ and that

$$[f]_{\mathscr{B}} = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

Let $n \in \mathbb{N}$. Since $f^{(n)} = \psi^n(f)$, and since the matrix of ψ in the basis $\mathscr{B}$ is A, we have

$$\left[f^{(n)}\right]_{\mathscr{B}} = \left[\psi^n(f)\right]_{\mathscr{B}} = A^n[f]_{\mathscr{B}} = A^n\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

From the form of A^n that we obtained in Part I, we conclude that

$$\left[f^{(n)}\right]_{\mathscr{B}} = \begin{pmatrix}2^{n-1}n + 2^{n-2}n(n-1)\\2^n(n+1)\\2^n\end{pmatrix} = \begin{pmatrix}2^{n-2}(2n + n(n-1))\\2^n(n+1)\\2^n\end{pmatrix} = \begin{pmatrix}2^{n-2}n(n+1)\\2^n(n+1)\\2^n\end{pmatrix} = 2^{n-2}\begin{pmatrix}n(n+1)\\4(n+1)\\4\end{pmatrix},$$

hence

$$\forall x \in \mathbf{R},\ f^{(n)}(x) = 2^{n-2}\left(n(n+1) + 4(n+1)x + 4x^2\right)\mathrm{e}^{2x}.$$