Mathematics

Final Exam

Duration: 2h42+18'

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1 (*1.5 pts*). Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$\begin{cases} a_0 = 0 \\ \forall n \in \mathbb{N},\ a_{n+1} = a_n + 2^n + n. \end{cases}$$

Define the sequence $(u_n)_{n\in\mathbb{N}}$ as

$$\forall n \in \mathbb{N},\ u_n = a_{n+1} - a_n.$$

For $N \in \mathbb{N}$, determine an explicit expression of

$$S_N = \sum_{n=0}^{N} u_n,$$

and deduce an explicit expression of the sequence $(a_n)_{n\in\mathbb{N}}$.

Exercise 2 (*2.5 pts*). Let $A \in M_3(\mathbb{R})$ be the following matrix:

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 4 & 0 \\ -1 & 1 & 3 \end{pmatrix}.$$

Show that A is diagonalizable, and determine a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Deduce the general solution of the following differential system:

$$(S)\qquad \begin{cases} x'(t) = 3x(t) + y(t) - z(t) \\ y'(t) = 4y(t) \\ z'(t) = -x(t) + y(t) + 3z(t). \end{cases}$$

You will write the system (S) in matrix form, i.e., $X'(t) = AX(t)$, where $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, and use the fact that $A = PDP^{-1}$.

You might want to set $U(t) = P^{-1}X(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}$, and write a (simple) system in terms of u, v and w that is equivalent to System (S), solve this system, and deduce the expression of x, y and z.

Exercise 3 (*8 pts*).

Part I – *Catalan's constant*

Let F be the function defined by

$$F : \mathbb{R}_+^* \longrightarrow \mathbb{R}$$
$$x \longmapsto \int_1^x \frac{\ln(t)}{1+t^2}\,dt.$$

1. Determine the sign of F.
2. Show that F is of class C^1 and determine a simple expression of F'.
3. Use an appropriate substitution to show that

$$\forall x \in \mathbb{R}_+^*,\ F(x) = F\left(\frac{1}{x}\right).$$

4. The goal of this question is to show that F possesses an extension by continuity (from the right at 0). We define the function

$$\varphi : \mathbb{R}^* \longrightarrow \mathbb{R}$$
$$t \longmapsto \frac{\arctan t}{t}.$$

   a) Show that φ possesses an extension by continuity at 0. We denote by $\tilde{\varphi}$ this extension by continuity.

   b) Deduce that the mapping

$$\Phi : \mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longmapsto \int_0^x \tilde{\varphi}(t)\,dt$$

   is continuous.

   c) Show that

$$\forall x \in \mathbb{R}_+^*,\ F(x) = \arctan(x)\ln(x) - \Phi(x) + \Phi(1).$$

   d) Deduce that F possesses an extension by continuity (from the right) at 0. We denote by $\tilde{F}$ the extension by continuity of F on $\mathbb{R}_+$. Determine the value of $\tilde{F}(0)$ in terms of Φ.

5. Show that $\tilde{F}$ is not differentiable (from the right) at 0.
6. Use the Mean Value Theorem (MVT2) to show that

$$\frac{1}{2} \leq \tilde{F}(0) \leq 1$$

The value of $\tilde{F}(0)$ is know as Catalan's constant (and not Catalin's constant). The next part of this exercise is devoted to the determination of a numerical approximation of $\tilde{F}(0)$.

Part II – *Approximation of Catalan's constant*

1. Let $k \in \mathbb{N}$ and $x \in \mathbb{R}_+^*$. Compute the value of the integral

$$I_k(x) = \int_1^x t^k \ln(t)\, dt$$

and deduce that

$$\lim_{x \to 0^+} I_k(x) = \lim_{x \to 0^+} \int_1^x t^k \ln(t)\, dt = \frac{1}{(k+1)^2}.$$

2. Let $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Show that

$$\left| \frac{1}{1+t^2} - \sum_{k=0}^{n} (-1)^k t^{2k} \right| \leq t^{2n+2}.$$

Keyword hint: sum of the terms of a geometric sequence.

3. Deduce that F(x)

$$\forall n \in \mathbb{N},\ \forall x \in (0,1],\quad \left| \int_1^x \frac{\ln(t)}{1+t^2}\, dt - \sum_{k=0}^{n} (-1)^k I_{2k}(x) \right| \leq I_{2n+2}(x),$$

and conclude that

$$\forall n \in \mathbb{N},\ \left| \tilde{F}(0) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2} \right| \leq \frac{1}{(2n+3)^2}.$$

4. Using Maple,[1] we computed:

$$\sum_{k=0}^{500} \frac{(-1)^k}{(2k+1)^2} = 0.9159660921817150075488240007091519696956849028491453875650482846316103923254744517\ldots$$

Deduce a lower bound and an upper bound of $\tilde{F}(0)$, as well as an approximation of $\tilde{F}(0)$ with as many correct decimal places as possible.

[1] the command is: `evalf[100](add((-1)^k/(2*k+1)^2,k=0..500))` but I truncated the output, so that we're sure that all the digits written are correct. No electrons were harmed in the process.

Exercise 4 (*6 pts*).

Part I – *Powers of a non-diagonalizable matrix*

Let $A \in \mathcal{M}_3(\mathbb{R})$ be the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

1. What are the eigenvalues of A and their multiplicities? deduce that A is not diagonalizable.

2. We define the matrix N as:

$$N = A - 2I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Compute N^2 and N^3, and for $k \geq 3$, deduce the value of N^k.

3. Let $n \geq 2$. Use the Binomial Theorem[2] to determine an explicit expression of A^n. Is your expression valid for $n = 1$? for $n = 0$?

Part II – *Application to the higher order derivatives of a function*

Let $E = C^\infty(\mathbb{R})$ be the real vector space of functions of class C^∞ on $\mathbb{R}$. we define the following vectors of E:

$$f_0 : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & e^{2x}, \end{array} \qquad f_1 : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & xe^{2x}, \end{array} \qquad f_2 : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^2e^{2x}, \end{array}$$

and we define the following subspace of E:

$$F = \operatorname{Span}\{f_0, f_1, f_2\}.$$

1. Show that the family $\mathscr{B} = (f_0, f_1, f_2)$ is a basis of F. What is the dimension of F?

2. Let ψ be the following mapping:

$$\psi : \begin{array}{ccc} F & \longrightarrow & F \\ f & \longmapsto & f'. \end{array}$$

   Show that ψ is a well-defined endomorphism of F.

3. Determine the matrix $[\psi]_{\mathscr{B}}$ of ψ in the basis $\mathscr{B}$.

4. Let $n \in \mathbb{N}$. Use the result of Part I to determine an explicit expression for the n-th derivative of the function

$$f : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \left(x + x^2\right)e^{2x}. \end{array}$$

[2] and justify that you're allowed to apply it!