INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Solution of Math Test #3

January 6, 2017

Romaric Pujol, romaric.pujol@insa-lyon.fr

Exercise 1.

1. Let $x \in \mathbb{R} \setminus \pi/6$. Then:

$$\frac{2\sin(x)-1}{6x-\pi} = \frac{1}{3}\frac{\sin(x)-\sin(\pi/6)}{x-\pi/6} \underset{x\to\pi/6}{\longrightarrow} \frac{1}{3}\cos(\pi/6) = \frac{1}{2\sqrt{3}}.$$

Let $x \in [0,1) \cup (1,+\infty)$. Then:

$$\frac{\sqrt{x}-1}{x-1} = \frac{1}{\sqrt{x}+1} \underset{x\to 1}{\longrightarrow} \frac{1}{2}.$$

2. Let $n \in \mathbb{N}$. Then

$$u_n = \frac{n-2}{n+1} = \frac{1-2/n}{1+1/n} \underset{n\to+\infty}{\longrightarrow} 1,$$

$$u_n^n = \left(\frac{1-2/n}{1+1/n}\right)^n = \frac{(1-2/n)^n}{(1+1/n)^n} \underset{e^{-2}}{\longrightarrow} e^1 = e^{-3}.$$

Exercise 2.

1. Let $x \in \mathbb{R}^*$. Then

$$f(x) = \frac{\sin(x)}{x}\frac{\sin(x)}{1+|x|}.$$

We know that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

and that

$$\lim_{x\to 0}\sin(x) = 0$$

hence, by the elementary operations on limits,

$$\lim_{x\to 0} f(x) = 1 \times \frac{0}{1+0} = 0 \in \mathbb{R}.$$

Hence f admits an extension by continuity at 0, namely

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} f(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

2. Let $x \in \mathbb{R}^*$. Then

$$\frac{\tilde{f}(x)-\tilde{f}(0)}{x-0} = \frac{\sin^2(x)}{x^2}\frac{1}{1+|x|} \underset{x\to 0}{\longrightarrow} 1 \times \frac{1}{1+0} = 1.$$

Hence the function $\tilde{f}$ is differentiable at 0 and $\tilde{f}'(0) = 1$. Hence an equation of the tangent line to the graph of $\tilde{f}$ at $(0, \tilde{f}(0))$ is:

$$y = x.$$

Exercise 3.

1. We know that arcsin and arccos are defined only on $[-1,1]$. Now, for $x \in [-1,1]$,

$$\arccos(x) = 0 \iff x = 1,$$

hence the given expression is defined for $x \in D = [-1,1)$.

2. Let $x \in D$. Then

$$f(x) = \frac{\arcsin(x)}{\arccos(x)} = \frac{\pi/2 - \arccos(x)}{\arccos(x)} = \frac{\pi}{2\arccos(x)} - 1.$$

3. We know that arccos is positive and decreasing on D. Hence $1/\arccos$ is increasing on D, and we conclude that f is increasing.

4. We know that arccos and arcsin are continuous. f is hence a quotient of two continuous functions, hence f is continuous.

5. We know that f is *continuous* and increasing. Hence, by (a corollary of) the intermediate value theorem

$$J = f(D) = f([-1,1)) = \left[f(-1), \lim_{x\to 1^-} f(x)\right).$$

Now,

$$f(-1) = \frac{\arcsin(-1)}{\arccos(-1)} = \frac{-\pi/2}{\pi} = -\frac{1}{2}.$$

Moreover, by continuity of arccos and arcsin at 1,

$$\lim_{x\to 1^-} \arccos(x) = \arccos(1) = 0, \qquad \lim_{x\to 1^-} \arcsin(x) = \arcsin(1) = \pi/2 > 0.$$

Moreover,

$$\forall x \in [-1,1),\ \arccos(x) > 0,$$

hence

$$\lim_{x\to 1^-} f(x) = +\infty.$$

Hence

$$J = f(D) = [-1/2, +\infty).$$

6. Since f is increasing, g is also increasing, hence g is injective. Since the codomain of g coincides with the range of g (that's how J was constructed), we conclude that g is surjective too, hence g is a bijection.

7. Let $x \in D$ and $y \in [-1/2, +\infty)$. Then

$$\begin{aligned} f(x) = y &\iff \frac{\pi}{2\arccos(x)} - 1 = y \\ &\iff \frac{\pi}{2\arccos(x)} = y + 1 \\ &\iff \frac{2\arccos(x)}{\pi} = \frac{1}{y+1} \\ &\iff \arccos(x) = \frac{\pi}{2(y+1)} \\ &\iff x = \cos\left(\frac{\pi}{2(y+1)}\right). \end{aligned}$$

Hence

$$\begin{aligned} g^{-1} : [-1/2, +\infty) &\longrightarrow [-1,1) \\ y &\longmapsto \cos\left(\frac{\pi}{2(y+1)}\right). \end{aligned}$$

Exercise 4.

1. We know that

$$\forall n \in \mathbb{N},\ -1 \le u_n \le 1,$$

hence $-1 \le \ell \le 1$.

2. The sequence $(u_{2n})_{n\in\mathbb{N}}$ is the even subsequence of $(u_n)_{n\in\mathbb{N}}$ and hence converges to ℓ too.

3. Let $n \in \mathbb{N}$. Then:

$$u_{2n} = \cos(2n) = 2\cos^2(n) - 1 = 2u_n^2 - 1.$$

Now, by the elementary operations on limits and the previous question, we conclude:

$$\ell = 2\ell^2 - 1,$$

hence $2\ell^2 - \ell - 1 = 0$.

4.

$$2\ell^2 - \ell - 1 = 2\left(\ell^2 - \frac{\ell}{2} - \frac{1}{2}\right) = 2\left(\left(\ell - \frac{1}{4}\right)^2 - \frac{9}{16}\right) = 2(\ell - 1)\left(\ell + \frac{1}{2}\right).$$

Hence the possible values for ℓ are 1 and $-1/2$.

5. Let $n \in \mathbb{N}$. Then:

$$u_{n+1} = \cos(n+1) = \cos(n)\cos(1) - \sin(n)\sin(1) = \cos(1)u_n - \sin(1)v_n,$$

hence

$$v_n = \frac{1}{\sin(1)}\big(\cos(1)u_n - u_{n+1}\big).$$

We know that $\lim_{n\to+\infty} u_{n+1} = \ell$ hence, by the elementary operations on limits, $(v_n)_{n\in\mathbb{N}}$ converges to ℓ' with

$$\ell' = \frac{\cos(1) - 1}{\sin(1)}\ell.$$

6. We know, by the Pythagorean theorem, that

$$\forall n \in \mathbb{N},\ u_n^2 + v_n^2 = 1,$$

hence, by the elementary operations on limits,

$$\ell^2 + \ell'^2 = 1.$$

7. From Question 5 we also have

$$\ell'^2 = \frac{\big(\cos(1) - 1\big)^2}{\sin^2(1)}\ell^2,$$

hence

$$\sin^2(1)\ell'^2 = \big(\cos^2(1) - 2\cos(1) + 1\big)\ell^2.$$

Using the fact that $\ell'^2 = 1 - \ell^2$ we obtain:

$$\sin^2(1)(1 - \ell^2) = \big(\cos^2(1) - 2\cos(1) + 1\big)\ell^2,$$

i.e.,

$$\sin^2(1) = 2\big(1 - \cos(1)\big)\ell^2,$$

i.e.,

$$\big(1 - \cos^2(1)\big) = 2\big(1 - \cos(1)\big)\ell^2,$$

i.e.,

$$\big(1 - \cos(1)\big)\big(1 + \cos(1)\big) = 2\big(1 - \cos(1)\big)\ell^2,$$

i.e.,

$$1 + \cos(1) = 2\ell^2.$$

We now investigate the two cases for ℓ:

- $\ell = 1$ yields $\cos(1) = 1$ which is impossible,
- $\ell = -1/2$ yields $\cos(1) = -1/2$ which is impossible too.

We hence conclude that the limits of the sequence $(u_n)_{n\in\mathbb{N}}$ doesn't exist.

Exercise 5.

1. We proceed by induction:

   - By assumption, $u_0 = a > 0$ and $v_0 = b > 0$.
   - Assume that, for some $n \in \mathbb{N}$, one has $u_n > 0$ and $v_n > 0$. Then, $u_n v_n > 0$ and hence $u_{n+1} = \sqrt{u_n v_n} > 0$, and $u_n + v_n > 0$ and hence $v_{n+1} = (u_n + v_n)/2 > 0$.

2. We proceed by induction:

- By assumption, $u_0 = a < b = v_0$.
- Assume that $u_n < v_n$ for some $n \in \mathbb{N}$. Then

$$v_{n+1} - u_{n+1} = \frac{u_n + v_n}{2} - \sqrt{u_n v_n} = \frac{u_n - 2\sqrt{u_n v_n} + v_n}{2} = \frac{\left(\sqrt{u_n} - \sqrt{v_n}\right)^2}{2}.$$

Since $u_n < v_n$ we have $u_n \neq v_n$, hence $\sqrt{u_n} \neq \sqrt{v_n}$, hence $\sqrt{u_n} - \sqrt{v_n} \neq 0$, and we conclude that $v_{n+1} - u_{n+1} > 0$.

3. Let $n \in \mathbb{N}$. Then:

$$v_{n+1} - v_n = \frac{u_n + v_n}{2} - v_n = \frac{u_n - v_n}{2} < 0$$

by Question 2. Hence the sequence $(v_n)_{n\in\mathbb{N}}$ is decreasing.

4. By Question 1, 0 is a lower bound of the sequence $(v_n)_{n\in\mathbb{N}}$. Hence $(v_n)_{n\in\mathbb{N}}$ is a decreasing sequence, bounded from below hence, by the Monotone Limit Theorem, the sequence $(v_n)_{n\in\mathbb{N}}$ is convergent.

5. For $n \in \mathbb{N}$ we have:

$$u_n = 2v_{n+1} - v_n.$$

We know that $\lim_{n\to+\infty} v_{n+1} = \ell$ too hence, by the elementary operations on limits,

$$\lim_{n\to+\infty} u_n = 2\ell - \ell = \ell.$$

6. We now show that the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing: let $n \in \mathbb{N}$. Then

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{u_n v_n}}{u_n} = \sqrt{\frac{v_n}{u_n}} > 1 .$$

by Question 2, hence the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing. We hence have:

- the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing,
- the sequence $(v_n)_{n\in\mathbb{N}}$ is decreasing,
- by the elementary operations on limits, $\lim_{n\to+\infty} u_n - v_n = \ell - \ell = 0$.

We hence conclude that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent sequences.

7. Let $n \in \mathbb{N}$. We already know, by Question 2, that $v_{n+1} - u_{n+1} > 0$. Now, as computed in Question 2,

$$v_{n+1} - u_{n+1} = \frac{u_n + v_n}{2} - \sqrt{u_n v_n} = \frac{\left(\sqrt{v_n} - \sqrt{u_n}\right)^2}{2} = \frac{\left(\left(\sqrt{v_n} - \sqrt{u_n}\right)\left(\sqrt{v_n} + \sqrt{u_n}\right)\right)^2}{2\left(\sqrt{v_n} + \sqrt{u_n}\right)^2} = \frac{(v_n - u_n)^2}{2\left(\sqrt{v_n} + \sqrt{u_n}\right)^2}.$$

8. We know that the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing, hence $u_n > u_0 = a$ hence $\sqrt{u_n} > \sqrt{a}$. Moreover, since $u_n < v_n$, we also have $\sqrt{v_n} > \sqrt{u_n} > \sqrt{a}$. Hence $\sqrt{u_n} + \sqrt{v_n} > 2\sqrt{a}$ hence

$$\frac{1}{\sqrt{u_n} + \sqrt{v_n}} < \frac{1}{2\sqrt{a}},$$

hence

$$\frac{1}{\left(\sqrt{u_n} + \sqrt{v_n}\right)^2} < \frac{1}{4a},$$

and we conclude that

$$0 < v_{n+1} - u_{n+1} < \frac{(v_n - u_n)^2}{8a}.$$