INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S2 — Math Test #5 — 2h

May 12, 2017

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The goal of this exercise is to compute the value of

$$I = \int_0^{\pi/2} \frac{\sin t}{1+\sin t}\,\mathrm{d}t.$$

1. Determine the partial fraction decomposition of

$$R = \frac{X}{(1+X)^2\left(1+X^2\right)}.$$

2. Determine an antiderivative F of the function

$$f : \begin{array}{ccc} (-1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x}{(1+x)^2\left(1+x^2\right)}. \end{array}$$

3. Use the substitution[2] $x = \tan(t/2)$ to show that

$$\int_0^{\pi/2} \frac{\sin t}{1+\sin t}\,\mathrm{d}t = 4\int_a^b f(x)\,\mathrm{d}x$$

with appropriate real numbers a and b that you will determine.

4. Deduce the value of I.

Exercise 2.

1. Define the function g as

$$g : \begin{array}{ccc} (-\infty,1) & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{t}{\sqrt{1-t}}. \end{array}$$

Determine an antiderivative G of g. Hint: $t = -(1-t)+1$.

2. Define the function

$$f : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \sqrt{t}\arccos\left(\sqrt{t}\right). \end{array}$$

a) Briefly explain why f is well-defined.

b) Use an integration by parts to determine an antiderivative F_1 of f on $(0,1)$. The result of Question 1 will be useful.

c) Deduce an antiderivative F of f on $[0,1]$.

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[2] You may use, without any justifications, the following result:

$$\forall t \in (-\pi,\pi),\ \sin(t) = \frac{2\tan(t/2)}{1+\tan^2(t/2)}.$$

Exercise 3. We define the function[3]

$$\begin{aligned} f\ :\ \mathbb{R}^* &\longrightarrow \mathbb{R} \\ x &\longmapsto \int_{1/(2x)}^{1/x} \frac{\sin(t)}{t^2}\,dt. \end{aligned}$$

1. Use a simple substitution to show that f is even.
2. Show that f is of class C^1 (or, even better, of class C^∞) and determine an explicit expression of f'.
3. Use the triangle inequality to show that

$$\forall x \in \mathbb{R}_+^*,\ \left|f(x)\right| \le x,$$

and deduce that f possesses an extension by continuity at 0.

4. a) Show, using the Taylor–Lagrange formula, that

$$\forall t \in \mathbb{R},\ \left|\sin t - t + \frac{t^3}{6}\right| \le \frac{t^4}{24}.$$

b) Deduce that

$$\forall x \in \mathbb{R}_+^*,\ \left|f(x) - \int_{1/(2x)}^{1/x}\left(\frac{1}{t} - \frac{t}{6}\right)\mathrm{d}t\right| \le \int_{1/(2x)}^{1/x}\frac{t^2}{24}\,\mathrm{d}t.$$

c) For $x \in \mathbb{R}_+^*$, compute the value of the two integrals that appear in the previous question.

d) Deduce that

$$f(x) \underset{x\to+\infty}{=} \ln(2) - \frac{1}{16x^2} + o\left(\frac{1}{x^2}\right).$$

Exercise 4. Let E be a vector space over $\mathbb{K}$ and let $\mathscr{B} = (u_1,\dots,u_n)$ be a basis of E. Let $\alpha_1,\dots,\alpha_n \in \mathbb{K}$ and let $w \in E$. Recall (without any justifications) what the following statement means:

$$[w]_{\mathscr{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Exercise 5. We consider the following linear system:

$$(S)\qquad \begin{cases} x + 2y + z = a \\ x + 3y + 3z = b \\ -2x + y + 8z = c \end{cases}$$

with unknowns $x,y,z \in \mathbb{R}$ and constant terms $a,b,c \in \mathbb{R}$.

1. Perform the Gaussian descent to obtain a triangular system that is equivalent to System (S). What is the rank of System (S)?
2. Deduce a necessary and sufficient condition on the numbers a, b and c for System (S) to admit solutions.
3. When this condition is fulfilled, find the solutions of System (S).
4. We consider the following vectors of $\mathbb{R}^3$:

$$u = (1,1,-2), \qquad v = (2,3,1), \qquad w = (1,3,8).$$

Is the vector $p = (1,1,1)$ a linear combination of the family (u,v,w)? Is the family (u,v,w) a basis of $\mathbb{R}^3$?

Exercise 6. Let E and F be two vector spaces over $\mathbb{K}$, and let $f : E \to F$ be a linear map.

1. Recall the definition of $\operatorname{Ker} f$.
2. Show that $\operatorname{Ker} f$ is a linear subspace of E.
3. Show that

$$f \text{ is injective} \iff \operatorname{Ker} f = \{0_E\}.$$

[3] You don't need to justify that f is well-defined