INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Solution of Math Test #6

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Exercise 1.

1.
$$\operatorname{rk}(A-I_3)=\operatorname{rk}\begin{pmatrix}-1&1&0\\0&-1&1\\-2&1&1\end{pmatrix}\underset{C_2\leftarrow C_2+C_1}{=}\operatorname{rk}\begin{pmatrix}-1&0&0\\0&-1&1\\-2&-1&1\end{pmatrix}=2.$$

Hence the matrix $A-I_3$ is not invertible, hence 1 is an eigenvalue of A.

2. If we set $X_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ we have $AX_1=X_1$, hence X_1 is an eigenvector of A associated with the eigenvalue 1.

3.
$$\chi_A(\lambda)=\det(A-\lambda I_3)=\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\-2&1&2-\lambda\end{vmatrix}\underset{C_1\leftarrow C_1+C_2+C_3}{=}\begin{vmatrix}1-\lambda&1&0\\1-\lambda&-\lambda&1\\1-\lambda&1&2-\lambda\end{vmatrix}\underset{\substack{R_2\leftarrow R_2-R_1\\R_3\leftarrow R_3-R_1}}{=}\begin{vmatrix}1-\lambda&1&0\\0&-1-\lambda&1\\0&0&2-\lambda\end{vmatrix}$$
$$=-(\lambda-1)(\lambda+1)(\lambda-2)$$

Hence the eigenvalues of A are:

- 1 of multiplicity 1,
- -1 of multiplicity 1,
- 2 of multiplicity 1.

4. We already have an eigenvector of A associated with the eigenvalue 1. We now determine an eigenvector of A associated with the eigenvalue -1:

$$E_{-1}:\begin{cases}x+y=0\\y+z=0\\-2x+y+3z=0\end{cases}\underset{R_3\leftarrow R_3+2R_1}{\iff}\begin{cases}x+y=0\\y+z=0\\3y+3z=0\end{cases}\iff\begin{cases}x=z\\y=-z\\z=z.\end{cases}$$

We hence choose $X_{-1}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ as an eigenvector of A associated with -1.

Similarly for the eigenvalue 2:

$$E_2:\begin{cases}-2x+y=0\\-2y+z=0\\-2x+y=0\end{cases}\iff\begin{cases}x=z/4\\y=z/2\\z=z.\end{cases}$$

We hence choose $X_2=\begin{pmatrix}1\\2\\4\end{pmatrix}$ as an eigenvector of A associated with 2.

Finally, we set $P=\begin{pmatrix}1&1&1\\1&-2&2\\1&1&4\end{pmatrix}$. Since the columns of P consist of three eigenvectors of A associated with distinct eigenvalues, we know that P is invertible. If we set $D=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&2\end{pmatrix}$ we have $A=PDP^{-1}$.

Exercise 2.

1.
$$\operatorname{rk}(B-I_3)=\operatorname{rk}\begin{pmatrix}-1&1&0\\0&-1&1\\2&-5&3\end{pmatrix}\underset{C_2\leftarrow C_2+C1}{=}\operatorname{rk}\begin{pmatrix}-1&0&0\\0&-1&1\\2&-3&3\end{pmatrix}=2.$$

2.

$$\chi_B(\lambda) = \det(B - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} \underset{C_1 \leftarrow C_1 + C_2 + C_3}{=} \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1-\lambda & -\lambda & 1 \\ 1-\lambda & -5 & 4-\lambda \end{vmatrix} \underset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1}}{=} \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & -6 & 4-\lambda \end{vmatrix}$$

$$\begin{aligned} &= (1-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ -6 & 4-\lambda \end{vmatrix} = (1-\lambda)\big((-1-\lambda)(4-\lambda)+6\big) = (1-\lambda)\big(\lambda^2 - 3\lambda + 2\big) \\ &= (1-\lambda)(\lambda-1)(\lambda-3) \\ &= -(\lambda-1)^2(\lambda-3). \end{aligned}$$

Hence the eigenvalues of B are:

- 1 of multiplicity 2,
- 2 of multiplicity 1.

Now, from Question 1 and by the Rank–Nullity Theorem we know that the dimension of the eigenspace of B associated with 1 is $\dim E_1 = 3 - \operatorname{rk}(B - I_3) = 1$. We notice that multiplicity$(1) = 2 \neq \dim E_1 = 1$, hence B is not diagonalizable.

3. a)
   - Clearly, $BU = U$ (obtained as the sum of all three columns of B), hence U is an eigenvector of B associated with 1.
   - Also, $BW = \begin{pmatrix} 2 \\ 4 \\ 2-10+16 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = 2W$, hence W is an eigenvector of B associated with the eigenvalue 2.
   - Finally, $BV = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = U + V$, hence $(B - I_3)V = U$.

   b) Let $x, y, z, a, b, c \in \mathbb{R}$. Then:

$$B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \iff \begin{cases} x - y + z = a \\ x + 2z = b \\ x + y + 4z = c \end{cases} \underset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1}}{\iff} \begin{cases} x - y + z = a \\ y + z = b - a \\ 2y + 3z = c - a \end{cases} \underset{R_3 \leftarrow R_3 - 2R_2}{\iff} \begin{cases} x - y + z = a \\ y + z = b - a \\ z = a - 2b + c \end{cases}$$

$$\iff \begin{cases} x = -2a + 5b - 2c \\ y = -2a + 3b - c \\ z = a - 2b + c \end{cases}$$

   Hence P is invertible and

$$P^{-1} = \begin{pmatrix} -2 & 5 & -2 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

   c) Since the columns of P are U, V and W we have:

$$BP = \begin{pmatrix} | & | & | \\ BU & BV & BW \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ U & U+V & 2W \\ | & | & | \end{pmatrix}.$$

   Now,

$$P\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = U, \qquad P\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = U + V, \qquad P\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 2W,$$

   hence

$$P^{-1}U = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad P^{-1}(U+V) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad P^{-1}(2W) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

   so that

$$T = P^{-1}BP = \begin{pmatrix} | & | & | \\ P^{-1}U & P^{-1}(U+V) & P^{-1}(2W) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Let $n \in \mathbb{N}$ with $n \geq 2$.

   a) $N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ hence, $N^n = 0_{M_2(\mathbb{R})}$.

   b)
$$DN = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad ND = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

   hence $DN = ND$, i.e., N and D commute. Now, $T = D + N$ hence, by the Binomial Theorem,
$$\begin{aligned} T^n &= (D+N)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} D^{n-k} N^k \\ &= D^n N^0 + \binom{n}{1} D^{n-1} N^1 \qquad \textit{since } \forall k \geq 2,\ N^k = 0_{M_2(\mathbb{R})} \\ &= D^n + nD^{n-1}N \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} + n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} + n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}. \end{aligned}$$

   c) Now $B^n = PT^nP^{-1}$, hence
$$\begin{aligned} B^n &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} -2 & 5 & -2 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n-1 & 2^n \\ 1 & n & 2^{n+1} \\ 1 & n+1 & 2^{n+2} \end{pmatrix} \begin{pmatrix} -2 & 5 & -2 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2n + 2^n & 2 + 3n - 2^{n+1} & -1 - n + 2^n \\ -2n - 1 + 2^{n+1} & 5 + 3n - 2^{n+2} & -2 - n + 2^{n+1} \\ -2n - 4 + 2^{n+2} & 8 + 3n - 2^{n+3} & -3 - n + 2^{n+2} \end{pmatrix} \end{aligned}$$

Exercise 3.

1.
$$\operatorname{rk}(C - I_3) = \operatorname{rk} \begin{pmatrix} -2 & 4 & -2 \\ -4 & 8 & -4 \\ -8 & 16 & -8 \end{pmatrix} = 1$$

   (since all three columns are proportional).

2. Hence, since $C - I_3$ is non-invertible, 1 is an eigenvalue of C. Moreover, by the Rank–Nullity Theorem, the dimension of the eigenspace associated with the eigenvalue 1 is $\dim E_1 = 3 - \operatorname{rk}(C - I_3) = 2$. Hence we conclude that the multiplicity of 1 is at least 2.

3. We're missing one eigenvalue. To determine it we use the trace of C: $\operatorname{tr}(C) = 1 = 1 + 1 + \text{missing eigenvalue}$. Hence the other eigenvalue of C is -1. We conclude that the eigenvalues of C are:

   - 1 of multiplicity 2,
   - -1 of multiplicity 1.

   Since $\dim E_1 = 2 = \text{multiplicity}(1)$ and since -1 is of multiplicity 1, we conclude that C is diagonalizable.

Exercise 4.

1. Let $\lambda, \lambda' \in \mathbb{K}$ with $\lambda \neq \lambda'$. We show that E_λ and $E_{\lambda'}$ are independent by showing that $E_\lambda \cap E_{\lambda'} = \{0_E\}$: let $u \in E_\lambda \cap E_{\lambda'}$. Since $u \in E_\lambda$ we have $f(u) = \lambda u$ and since $u \in E_{\lambda'}$ we have $f(u) = \lambda' u$. Hence $f(u) = \lambda u = \lambda' u$ hence $(\lambda - \lambda')u = 0_E$. Since $\lambda \neq \lambda'$, we have $\lambda - \lambda' \neq 0$ and we must hence have $u = 0_E$. We conclude that $E_\lambda \cap E_{\lambda'} = \{0_E\}$.

2. a) Let $u = (x, y) \in E$. Then:

$$u \in E_4 \iff (f - 4\,\mathrm{id}_E)(u) = 0_E \iff (-3x - 3y, -3x - 3y) = (0, 0) \iff x + y = 0 \iff u = x(1, -1).$$

We conclude that $E_4 = \operatorname{Span}\{(1, -1)\} \neq \{0_E\}$ and that $\big((1, -1)\big)$ is a basis of E_4.

b) Let $\lambda \in \mathbb{R} \setminus \{4\}$. Let $u = (x, y) \in E$. Then:

$$u \in E_\lambda \iff (f - \lambda\,\mathrm{id}_E)(u) = 0_E \iff \big((1-\lambda)x - 3y, -3x + (1-\lambda)y\big) = (0, 0) \iff \begin{cases} -3x + (1-\lambda)y = 0 \\ (1-\lambda)x - 3y = 0 \end{cases}$$

$$\underset{R_2 \leftarrow R_2 + \frac{(1-\lambda)}{3} R_1}{\iff} \begin{cases} -3x + (1-\lambda)y = 0 \\ \left(\frac{1}{3}(1-\lambda)^2 - 3\right) y = 0 \end{cases} \underset{R_2 \leftarrow 3R_2}{\iff} \begin{cases} -3x + (1-\lambda)y = 0 \\ \left((1-\lambda)^2 - 9\right) y = 0 \end{cases}$$

$$\iff \begin{cases} -3x + (1-\lambda)y = 0 \\ (\lambda + 2)(\lambda - 4)y = 0 \end{cases} \underset{R_2 \leftarrow \frac{1}{\lambda - 4} R_2}{\iff} \begin{cases} -3x + (1-\lambda)y = 0 \\ (\lambda + 2)y = 0 \end{cases}$$

Hence the rank of this system is $\begin{cases} 1 & \text{if } \lambda = -2 \\ 2 & \text{otherwise.} \end{cases}$ Hence

$$E_\lambda \neq \{0_E\} \iff \lambda = -2,$$

and

$$u \in E_{-2} \iff -3x + 3y = 0 \iff x = y \iff u = x(1, 1).$$

Hence a basis of E_{-2} is $\big((1, 1)\big)$.

c) We know that E_4 and E_{-2} are independent. By Grassmann's Formula:

$$\dim(E_4 \oplus E_{-2}) = \dim E_4 + \dim E_{-2} = 1 + 1 = 2 = \dim E.$$

Hence, by the Inclusion–Equality Theorem, $E_4 \oplus E_{-2} = E$ and we conclude that E_4 and E_{-2} are complementary subspaces in E.

d) Set $\mathscr{B} = \big((1, -1), (1, 1)\big)$. Since E_4 and E_{-2} are complementary subspaces in E, we conclude that $\mathscr{B}$ is a basis of E. We have:

$$[f]_{\mathscr{B}} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Exercise 5.

1. a) $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{pmatrix}$.

b) $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix}$.

c) Let $(x, y, z) \in F$. Then

$$(f \circ g)(x, y, z) = f\big(g(x, y, z)\big) = f(x + y + z, 2x - y - z) = (3x, 3y + 3z, -x + 2y + 2z).$$

Hence $C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 3 \\ -1 & 2 & 2 \end{pmatrix}$.

d)

$$C = AB = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 3 \\ -1 & 2 & 2 \end{pmatrix}.$$

2. a)
- $f(u_1) = f(1, 1) = (2, 1, 0) = v_1 + v_2$ hence $\big[f(u_1)\big]_{\mathscr{C}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

- $f(u_2) = f(1,-1) = (0,3,2) = -3v_1 + v_2 + 2v_3$ hence $\big[f(u_1)\big]_{\mathscr{C}} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

Hence

$$A' = \begin{pmatrix} 1 & -3 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

- $g(v_1) = (1,2) = \frac{3}{2}u_1 - \frac{1}{2}u_2$ hence $\big[g(v_1)\big]_{\mathscr{B}} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$.
- $g(v_2) = (2,1) = \frac{3}{2}u_1 + \frac{1}{2}u_2$ hence $\big[g(v_1)\big]_{\mathscr{B}} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$.
- $g(v_3) = (3,0) = \frac{3}{2}u_1 + \frac{3}{2}u_2$ hence $\big[g(v_1)\big]_{\mathscr{B}} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$.

Hence

$$B' = \begin{pmatrix} 3/2 & 3/2 & 3/2 \\ -1/2 & 1/2 & 3/2 \end{pmatrix}.$$

b) Hence

$$C' = A'B' = \begin{pmatrix} 1 & -3 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3/2 & 3/2 & 3/2 \\ -1/2 & 1/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{pmatrix}.$$

Exercise 6.