INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S2 — Math Test #6 — 2h

June 12, 2017

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$.

1. Determine the rank of $A - I_3$ and deduce that 1 is an eigenvalue of A.
2. Find an eigenvector X_1 of A associated with the eigenvalue 1.
3. Compute the characteristic polynomial of A and deduce all the eigenvalues of A and their multiplicities.
4. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Exercise 2. Let $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$.

1. Determine the rank of $B - I_3$ and deduce that 1 is an eigenvalue of B.
2. Compute the characteristic polynomial of B and deduce all the eigenvalues of B and their multiplicities.
3. Deduce that the matrix B is not diagonalizable.
4. We define:

$$U = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad V = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

   a) Show that U and W are eigenvectors of B (associated with the eigenvalues you will specify), and show that $(B - I_3)V = U$.

   b) We set

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}.$$

   Show that P is invertible and determine P^{-1}.

   c) Determine the matrix $T = P^{-1}BP$. (Note: it's possible to determine T without explicitly performing the product).

5. We set

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Let $n \in \mathbb{N}$ with $n \geq 2$.

   a) Determine the value of N^n.

   b) Show that N and D commute and deduce the value of T^n. (The matrix T is the one you obtained in the previous Question).

   c) Determine B^n.

Exercise 3. Let $C = \begin{pmatrix} -1 & 4 & -2 \\ -4 & 9 & -4 \\ -8 & 16 & -7 \end{pmatrix}$.

1. Determine the rank of $C - I_3$.

2. Deduce that 1 is an eigenvalue of C. What can you say about its multiplicity?

3. Deduce all the eigenvalues of C and their multiplicities. Is C diagonalizable?

Exercise 4. Let E be a vector space over $\mathbb{K}$ and let $f \in L(E)$. For $\lambda \in \mathbb{K}$ we define

$$E_\lambda = \operatorname{Ker}(f - \lambda \operatorname{id}_E).$$

1. Let $\lambda, \lambda' \in \mathbb{K}$ with $\lambda \neq \lambda'$. Show that the subspaces E_λ and $E_{\lambda'}$ are independent.

2. Let f be the following endomorphism of $E = \mathbb{R}^2$:

$$f : \begin{array}{ccc} E & \longrightarrow & E \\ (x,y) & \longmapsto & (x-3y, -3x+y). \end{array}$$

   a) Show that $E_4 \neq \{0_E\}$ and determine a basis of E_4.

   b) Show that there exists a unique $\lambda \in \mathbb{R} \setminus \{4\}$ such that $E_\lambda \neq \{0_E\}$, and determine a basis of E_λ.

   c) Are the subspaces E_4 and E_λ complementary subspaces of E?

   d) Deduce a basis $\mathscr{B}$ of E such that the matrix $[f]_{\mathscr{B}}$ is diagonal (and explicit this matrix).

Exercise 5. Let

$$u_1 = (1,1), \quad u_2 = (1,-1), \quad v_1 = (1,0,0), \quad v_2 = (1,1,0), \quad v_3 = (1,1,1),$$

Let

$$\mathscr{B} = \big((1,1),(1,-1)\big) = (u_1,u_2) \quad \text{and} \quad \mathscr{C} = \big((1,0,0),(1,1,0),(1,1,1)\big) = (v_1,v_2,v_3).$$

You're given that $\mathscr{B}$ is a basis of $E = \mathbb{R}^2$ and that $\mathscr{C}$ is a basis of $F = \mathbb{R}^3$.
We moreover define the following two linear maps:

$$f : \begin{array}{ccc} E & \longrightarrow & F \\ (x,y) & \longmapsto & (x+y, 2x-y, x-y), \end{array} \qquad g : \begin{array}{ccc} F & \longrightarrow & E \\ (x,y,z) & \longmapsto & (x+y+z, 2x-y-z). \end{array}$$

1. a) Determine the matrix $A = [f]_{\mathrm{std}(R^2),\mathrm{std}(R^3)}$ of f in the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

   b) Determine the matrix $B = [g]_{\mathrm{std}(R^3),\mathrm{std}(R^2)}$ of g in the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

   c) Determine the expression of $f \circ g$ using the composition of the maps f and g, and deduce the matrix $C = [f \circ g]_{\mathrm{std}}$ of $f \circ g$ in the standard basis of $\mathbb{R}^3$.

   d) Recover the matrix C using the matrices A and B.

2. a) Determine the matrix $A' = [f]_{\mathscr{B},\mathscr{C}}$ and the matrix $B' = [g]_{\mathscr{C},\mathscr{B}}$.

   b) Deduce the matrix $C' = [f \circ g]_{\mathscr{C}}$.

Exercise 6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the endomorphism of $\mathbb{R}^2$ such that $f(1,0)$ and $f(0,1)$ are shown in Figure 2. Plot on Figure 2 the image by f of the house shown in Figure 1.

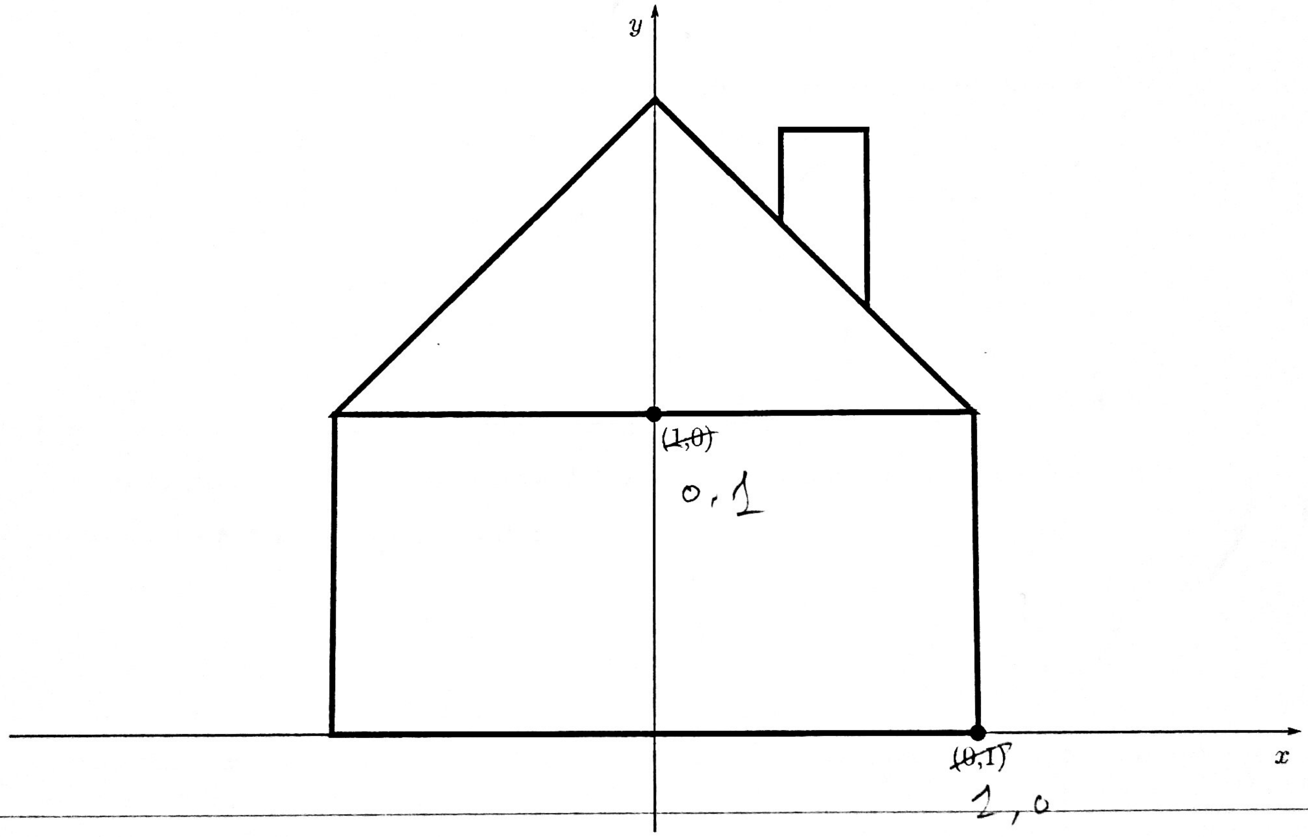

Figure 1 – Original house