INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Solution of Math Test #3

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Exercise 1.

1. By induction: the initial step is true since $u_1 = \sqrt{2}$. Assume that $0 < u_n < 2$ for some $n \geq 1$. Then $2 < 2 + u_n < 4$, and hence $\sqrt{2} < \sqrt{2+u_n} < 2$, and hence $0 < u_{n+1} < 2$.

2. We first show that the sequence $(u_n)_{n\geq 1}$ is increasing: let $n \geq 1$. Then

$$u_{n+1} - u_n = \sqrt{2+u_n} - u_n = \frac{2 + u_n - u_n^2}{\sqrt{2+u_n} + u_n} = -\frac{(u_n+1)(u_n-2)}{\sqrt{2+u_n}+u_n}.$$

Now since $u_n \in (0,2)$ we have:

$$u_n + 1 > 0, \qquad u_n - 2 < 0, \qquad \sqrt{2+u_n} + u_n > 0$$

hence $u_{n+1} - u_n > 0$ hence the sequence $(u_n)_{n\geq 1}$ is increasing.

Since the sequence $(u_n)_{n\geq1}$ is increasing and bounded from above, we conclude, by the Monotone Limit Theorem, that $(u_n)_{n\geq1}$ converges. We denote by ℓ the limit of $(u_n)_{n\geq1}$. We conclude from Question 1 that $0 \leq \ell \leq 2$. Since for all $n \in \mathbb{N}^*$, $u_{n+1} = \sqrt{2+u_n}$, taking the limit as $n \to +\infty$ (using the elementary operations on limits) yields:

$$\ell = \sqrt{2+\ell}.$$

Now,

$$\begin{aligned}
\ell = \sqrt{2+\ell} &\iff \ell^2 = 2 + \ell && \text{since } \ell \geq 0\\
&\iff \ell^2 - \ell - 2 = 0\\
&\iff (\ell+1)(\ell-2) = 0\\
&\iff \ell = 2 && \text{since } \ell + 1 \neq 0 \text{ since } \ell \geq 0.
\end{aligned}$$

Hence $\lim\limits_{n\to+\infty} u_n = 2$.

3. For $N \in \mathbb{N}^*$ we define the proposition

$$(P_N) \qquad \forall x \in \mathbb{R},\ \sin(x) = 2^N \sin\left(\frac{x}{2^N}\right)\prod_{n=1}^{N}\cos\left(\frac{x}{2^n}\right).$$

We show that for all $N \in \mathbb{N}^*$, Proposition P_N is true, by induction on N:

- Base case: let $x \in \mathbb{R}$. Then

$$\sin(x) = \sin\left(2\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2^1\sin\left(\frac{x}{2^1}\right)\prod_{n=1}^{1}\cos\left(\frac{x}{2^n}\right).$$

Hence P_1 is true.

- Inductive step: assume that P_N is true for some $N \geq 1$. Let $x \in \mathbb{R}$. Then:

$$\begin{aligned}
\sin(x) &= 2^N\sin\left(\frac{x}{2^N}\right)\prod_{n=1}^{N}\cos\left(\frac{x}{2^n}\right) && \text{by } P_N\\
&= 2^N\sin\left(2\frac{x}{2^{N+1}}\right)\prod_{n=1}^{N}\cos\left(\frac{x}{2^n}\right) = 2^N 2\sin\left(\frac{x}{2^{N+1}}\right)\cos\left(\frac{x}{2^{N+1}}\right)\prod_{n=1}^{N}\cos\left(\frac{x}{2^n}\right)\\
&= 2^{N+1}\sin\left(\frac{x}{2^{N+1}}\right)\prod_{n=1}^{N+1}\cos\left(\frac{x}{2^n}\right)
\end{aligned}$$

Hence P_{N+1} is true.

4. By induction: for $n = 1$ we have

$$\cos\left(\frac{\pi}{2^{1+1}}\right) = \cos(\pi/4) = \frac{\sqrt{2}}{2} = \frac{u_1}{2},$$

hence the base case is true. Assume that, for some $n \in \mathbb{N}^*$, one has $\cos(\pi/2^{n+1}) = u_n/2$. Then:

$$\begin{aligned}
\cos^2\left(\frac{\pi}{2^{n+2}}\right) &= \frac{1}{2}\left(1 + \cos\left(\frac{\pi}{2^{n+1}}\right)\right)\\
&= \frac{1}{2}\left(1 + \frac{u_n}{2}\right) && \text{by our induction hypothesis}\\
&= \frac{1}{4}(2 + u_n).
\end{aligned}$$

Now, since $\pi/2^{n+2} \in [-\pi/2, \pi/2]$, we know that $\cos(\pi/2^{n+2}) \geq 0$, hence

$$\cos^2\left(\frac{\pi}{2^{n+2}}\right) = \sqrt{\frac{1}{4}(2+u_n)} = \frac{\sqrt{2+u_n}}{2} = \frac{u_{n+1}}{2}.$$

5. Let $N \in \mathbb{N}^*$. From the previous questions we have:

$$1 = \sin\left(\frac{\pi}{2}\right) = 2^N\sin\left(\frac{\pi}{2^{N+1}}\right)\prod_{n=1}^{N}\cos\left(\frac{\pi}{2^{n+1}}\right) = 2^N\sin\left(\frac{\pi}{2^{N+1}}\right)\prod_{n=1}^{N}\frac{u_n}{2} = \frac{\sin\left(\frac{\pi}{2^{N+1}}\right)}{\frac{1}{2^N}}\prod_{n=1}^{N}\frac{u_n}{2}$$
$$= \frac{\pi}{2}\frac{\sin\left(\frac{\pi}{2^{N+1}}\right)}{\frac{\pi}{2^{N+1}}}\prod_{n=1}^{N}\frac{u_n}{2},$$

and we conclude:

$$\forall N \in \mathbb{N}^*,\ \frac{2}{\pi} = \frac{\sin\left(\frac{\pi}{2^{N+1}}\right)}{\frac{\pi}{2^{N+1}}}\prod_{n=1}^{N}\frac{u_n}{2}.$$

Now, $\lim\limits_{N\to+\infty}\frac{\pi}{2^{N+1}} = 0$, hence

$$\lim_{N\to+\infty}\frac{\sin\left(\frac{\pi}{2^{N+1}}\right)}{\frac{\pi}{2^{N+1}}} = 1,$$

hence

$$\frac{2}{\pi} = \lim_{N\to+\infty}\prod_{n=1}^{N}\frac{u_n}{2}.$$

Exercise 2.

1. By the product rule,

$$\forall x \in \mathbb{R},\ f'(x) = (x+1)e^x$$

and we conclude that

$$\forall x \in (1, +\infty),\ f'(x) > 0.$$

Since f is continuous on $[-1, +\infty)$ and for every point $x_0 \in [-1,+\infty)\setminus\{-1\}$, $f'(x_0) > 0$, we conclude that f is increasing on $[-1, +\infty)$.

2. Since the function f is increasing on $[-1, +\infty)$, we must have

$$\forall x \in [-1, +\infty),\ f(x) \geq f(-1) = -1/e.$$

Hence g is well defined, and since g is increasing, g is injective. Moreover, since g is increasing and continuous, by the Intermediate Value Theorem, we must have:

$$g([-1,+\infty)) = \left[g(-1), \lim_{x\to+\infty} g(x)\right).$$

Now it is clear from the elementary operations on limits that $\lim\limits_{x\to+\infty} g(x) = +\infty$, hence the range of g is $[-1/e, +\infty)$, hence g is also surjective.

3. a) We know that for all $x \in (-1,+\infty)$, $g'(x) = (x+1)e^x \neq 0$, hence W is differentiable on $g((-1,+\infty)) = (-1/e,+\infty)$.
Since $g'(-1) = 0$, we conclude that W is not differentiable at $g(-1) = -1/e$.

b) We know from the Inverse Function Rule that

$$\forall x \in (-1/e,+\infty),\ W'(xe^x) = \frac{1}{(x+1)e^x}.$$

Hence taking $x = 0 \in (-1/e,+\infty)$ in the previous proposition yields: $W'(0) = 1$.

Exercise 3.

1. In the following are given the domain and the range of the functions, as well as the domain of differentiability and the formula for their derivative:

$$\arcsin : [-1,1] \to [-\pi/2,\pi/2] \qquad \arcsin' : \begin{array}{rcl}(-1,1) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{\sqrt{1-x^2}}\end{array}$$

$$\arctan : \mathbb{R} \to (-\pi/2,\pi/2) \qquad \arctan' : \begin{array}{rcl}\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{1+x^2}\end{array}$$

$$\sinh : \mathbb{R} \to \mathbb{R} \qquad \sinh' : \begin{array}{rcl}\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \cosh(x)\end{array}$$

$$\tanh : \mathbb{R} \to (-1,1) \qquad \tanh' : \begin{array}{rcl}\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 1-\tanh^2(x) = \dfrac{1}{\cosh^2(x)}\end{array}$$

2.
   - tanh is defined on $\mathbb{R}$ and its range is $(-1,1)$; arcsin is defined on $[-1,1]$ (which contains the range of tanh), hence $\arcsin(\tanh(x))$ is defined for all $x \in \mathbb{R}$.
   - sinh is defined on $\mathbb{R}$ and arctan too, hence $\arctan(\sinh(x))$ is defined on $\mathbb{R}$.

   Conclusion, $D = \mathbb{R}$.

3. a) By the chain rule: tanh is differentiable on $\mathbb{R}$ and arcsin is differentiable on $\tanh(\mathbb{R}) \subset (-1,1)$, hence f is differentiable on $\mathbb{R}$ and for $x \in \mathbb{R}$,

$$f'(x) = \tanh'(x)\arcsin'(\tanh(x)) = \frac{1}{\cosh^2(x)}\frac{1}{\sqrt{1-\tanh^2(x)}} = \frac{1}{\cosh^2(x)}\frac{1}{\sqrt{\dfrac{1}{\cosh^2(x)}}}$$

$$= \frac{1}{\cosh^2(x)}\frac{1}{\dfrac{1}{\cosh(x)}} \qquad \textit{since } \cosh(x) \geq 1 > 0$$

$$= \frac{1}{\cosh(x)}.$$

b) By the chain rule: sinh is differentiable on $\mathbb{R}$ and arctan is differentiable on $\sinh(\mathbb{R}) \subset \mathbb{R}$, hence g is differentiable on $\mathbb{R}$ and for $x \in \mathbb{R}$,

$$g'(x) = \sinh'(x)\arctan'(\sinh(x)) = \cosh(x)\frac{1}{1+\sinh^2(x)} = \cosh(x)\frac{1}{\cosh^2(x)} = \frac{1}{\cosh(x)}.$$

c) Hence u is differentiable on $\mathbb{R}$ and $u' = f' - g'$ and hence

$$\forall x \in \mathbb{R},\ u'(x) = \frac{1}{\cosh(x)} - \frac{1}{\cosh(x)} = 0.$$

d) Since $\mathbb{R}$ is an interval, we conclude that u is constant. Since

$$u(0) = f(0) - g(0) = \arcsin(\tanh(0)) - \arctan(\sinh(0)) = \arcsin(0) - \arctan(0) = 0 - 0 = 0,$$

we conclude that u is the nil function on $\mathbb{R}$. Hence f and g are equal. We hence showed that

$$\forall x \in \mathbb{R},\ \arcsin(\tanh(x)) = \arctan(\sinh(x)).$$

Exercise 4.

1. Let $x \in \mathbb{R}_+^*$. Since the function exp is continuous on $[0,x]$ and differentiable on $(0,x)$, by the Mean Value Theorem, there exists $c \in (0,x)$ such that

$$\frac{e^x - e^0}{x-0} = e^c,$$

i.e., $e^x = 1 + xe^c$. Now since $c \in (0,x) \subset \mathbb{R}^*$, we have $e^c > 1$, hence $xe^c > x$ and hence $e^x > 1 + x$.

2. For $x \in \mathbb{R}_+^*$ (which is an open set), $f(x) = e^{x\ln(x)}$. Hence, by the chain rule and the product rule, f is differentiable on $\mathbb{R}_+^*$ and for $x \in \mathbb{R}_+^*$,

$$f'(x) = (\ln(x)+1)e^{x\ln(x)} = (1+\ln(x))x^x.$$

Let $x \in (0,1)$ (which is a punctured right sided neighborhood of 0. Then

$$\frac{f(x)-f(0)}{x-0} = \frac{e^{x\ln(x)}-1}{x} = \ln(x)\frac{e^{x\ln(x)}-1}{x\ln(x)}.$$

Now we know that

$$\lim_{x\to 0^+} x\ln(x) = 0 \qquad \text{and} \qquad \lim_{X\to 0}\frac{e^X-1}{X} = 1,$$

hence

$$\lim_{x\to 0^+}\frac{e^{x\ln(x)}-1}{x\ln(x)} = 1.$$

But since $\lim_{x\to 0^+}\ln(x) = -\infty$, we conclude, by the elementary operations on limits, that

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = -\infty \times 1 = -\infty.$$

Hence f is not differentiable at 0.

3) Let $x \in \mathbb{R}^*$ (which is a punctured neighborhood of 0). Then

$$\frac{g(x)-g(0)}{x-0} = \frac{\sqrt{|x|}\sin(x)\cos(1/x)}{x} = \frac{\sin(x)}{x}\sqrt{|x|}\cos(1/x).$$

Now, the function $x \mapsto \cos(1/x)$ is bounded on $\mathbb{R}^*$, and we have $\lim x \to 0\sqrt{|x|} = 0$, hence

$$\lim_{x\to 0}\sqrt{|x|}\cos(1/x) = 0.$$

Moreover, we know that $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x} = 1$, hence $\lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = 1 \times 0 = 0$, hence g is differentiable at 0 and $g'(0) = 0$.

Exercise 5.

1. Since f is differentiable on $[a,b]$, f is continuous on $[a,b]$, hence g is continuous on $(a,b]$. Now since f is differentiable at a, it is clear from the definition of a derivative that $\lim\limits_{x\to a^+} g(x) = f'(a) = g(a)$, hence g is also continuous at a.

2. We use the Intermediate Value Theorem:
   - g is continuous on $[a,b]$,
   - $g(a) = f'(a) < 0$,
   - $g(b) = \dfrac{f(b)-f(a)}{b-a} > 0$ since $f(b) > f(a)$ and $b > a$.

   Hence, by the Intermediate Value Theorem, there exists $p \in (a,b)$ such that $g(p) = 0$. From $g(p) = 0$ we conclude that $f(p) = f(a)$.

3. We know that
   - f is continuous on $[a,p]$ (since $[a,p] \subset [a,b]$ and f is differentiable on $[a,b]$),
   - f is differentiable on (a,p) (since $(a,p) \subset [a,b]$ and f is differentiable on $[a,b]$),
   - $f(p) = f(a)$ (by the previous question),

   by Rolle's Theorem, there exists $c \in (a,p)$ such that $f'(c) = 0$.