INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Solution of Math Test #4

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Exercise 1.

1. For $x \in \mathbb{R}^*$,

$$\left(e^x - x\right)^{1/x^2} = \exp\left(\frac{1}{x^2}\ln\left(e^x - x\right)\right).$$

Now,

$$e^x - x \underset{x\to 0}{\longrightarrow} 1,$$

hence

$$\ln\left(e^x - x\right) \underset{x\to 0}{\sim} e^x - 1 - x \underset{x\to 0}{=} \frac{x^2}{2} + o\left(x^2\right) \underset{x\to 0}{\sim} \frac{x^2}{2}.$$

Hence

$$\frac{1}{x^2}\ln\left(e^x - x\right) \underset{x\to 0}{\sim} \frac{1}{2} \underset{x\to 0}{\longrightarrow} \frac{1}{2},$$

hence

$$\lim_{x\to 0}\left(e^x - x\right)^{1/x^2} = e^{1/2} = \sqrt{e}.$$

2. Let $x \in \mathbb{R}$. In order to have the integral well-defined, we need the function $t \mapsto e^t/t$ to be (piecewise) continuous $[x, x+1]$, which is the case if and only if $x > 0$ or $x < -1$. Hence $D = (-\infty, -1) \cup (0, +\infty)$.

3.

4. Let $n \in \mathbb{N}^*$. By an integration by parts,

$$\begin{aligned}\int_0^{2\pi} f(t)\sin(nt)\,dt &= \left[-f(t)\frac{\cos(nt)}{n}\right]_{t=0}^{t=2\pi} + \frac{1}{n}\int_0^{2\pi} f'(t)\cos(nt)\,dt\\ &= -\frac{f(2\pi)}{n} + \frac{f(0)}{n} + \frac{1}{n}\int_0^{2\pi} f'(t)\cos(nt)\,dt\\ &= \frac{1}{n}\int_0^{2\pi} f'(t)\cos(nt)\,dt \qquad \textit{since } f(0) = f(2\pi) \textit{ since } f \textit{ is periodic of period } 2\pi.\end{aligned}$$

Now, by the triangle inequality,

$$\left|\int_0^{2\pi} f'(t)\cos(nt)\,dt\right| \le \int_0^{2\pi} \left|f'(t)\cos(nt)\right|dt \le \int_0^{2\pi}\left|f'(t)\right|dt.$$

Hence,

$$\left|\int_0^{2\pi} f(t)\sin(nt)\,dt\right| \le \frac{1}{n}\int_0^{2\pi}\left|f'(t)\right|dt \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence, by the Squeeze Theorem,

$$\lim_{n\to+\infty}\int_0^{2\pi} f(t)\sin(nt)\,dt = 0.$$

5. Let $x \in \mathbb{R}$.

$$\frac{x+2}{x^2+x+1} = \frac{1}{2}\frac{2x+1}{x^2+x+1} + \frac{3}{2}\frac{1}{x^2+x+1}.$$

Now,

$$\begin{aligned}\frac{1}{x^2+x+1} &= \frac{1}{(x+1/2)^2 + 3/4}\\ &= \frac{4}{3}\frac{1}{4/3\,(x+1/2)^2 + 1}\\ &= \frac{4}{3}\frac{1}{\left(2x/\sqrt{3} + 1/\sqrt{3}\right)^2 + 1}.\end{aligned}$$

Hence an antiderivative of $x \mapsto \dfrac{1}{x^2+x+1}$ is given by

$$x \mapsto \frac{2}{\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)$$

Hence an antiderivative of $x \mapsto \dfrac{x+2}{x^2+x+1}$ is given by

$$x \mapsto \frac{1}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\arctan\left(\frac{2x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right).$$

6. The function $g : t \mapsto f(t) - t^2$ is continuous on $[0,1]$ hence, by the Mean Value Theorem (MVT1) there exists $c \in [0,1]$ such that

$$g(c) = \frac{1}{1-0}\int_0^1 g(t)\,dt.$$

Now,

$$\int_0^1 g(t)\,dt = \int_0^1 \left(f(t) - t^2\right)dt = \int_0^1 f(t)\,dt - \int_0^1 t^2\,dt = \frac{1}{3} - \left[\frac{t^3}{3}\right]_{t=0}^{t=1} = \frac{1}{3} - \frac{1}{3} = 0.$$

Hence $g(c) = f(c) - c^2 = 0$.

Exercise 2.

1. Since the function sin is of class C^6 on $[0, 2/3]$ and 7 times differentiable on $(0, 2/3)$ there exists $c \in (0, 2/3)$ such that

$$\begin{aligned}\sin(2/3) &= \sum_{k=0}^{6}\frac{\sin^{(k)}(0)}{k!}\left(\frac{2}{3}\right)^k + \frac{\sin^{(7)}(c)}{5040}\left(\frac{2}{3}\right)^7\\ &= \frac{2}{3} - \frac{1}{6}\left(\frac{2}{3}\right)^3 + \frac{1}{120}\left(\frac{2}{3}\right)^5 - \frac{\cos(c)}{5040}\left(\frac{2}{3}\right)^7.\end{aligned}$$

Since $c \in (0, 2/3) \subset (0, \pi/2)$, $0 < \cos(c) < 1$, hence

$$\left(\frac{2}{3}\right)^7\frac{1}{5040} < -\frac{\cos(c)}{5040}\left(\frac{2}{3}\right)^7 < 0.$$

Hence

$$\frac{2}{3} - \frac{1}{6}\left(\frac{2}{3}\right)^3 + \frac{1}{120}\left(\frac{2}{3}\right)^5 - \frac{1}{5040}\left(\frac{2}{3}\right)^7 < \sin(2/3) < \frac{2}{3} - \frac{1}{6}\left(\frac{2}{3}\right)^3 + \frac{1}{120}\left(\frac{2}{3}\right)^5.$$

2. With the values given, we conclude that

$$\frac{2254}{3645} - \frac{8}{688905} < \sin(2/3) < \frac{2254}{3645}.$$

Now we also read that:

$$\frac{2254}{3645} < 0.618382 \qquad \text{and} \qquad \frac{2254}{3645} - \frac{8}{688904} > 0.618381 - 0.000012 = 0.618369.$$

Hence

$$0.618369 < \sin(2/3) < 0.618382,$$

hence we get the value of $\sin(2/3)$ correct to 4 decimal places:

$$\sin(2/3) = 0.6183\ldots$$

Exercise 3.

1. a) We know that

$$\sin(x) \underset{x\to 0}{=} x - \frac{x^3}{6} + o\left(x^3\right)$$

hence
$$\sin(x) - x \underset{x\to}{=} -\frac{x^3}{6} + o(x^3) \underset{x\to0}{\sim} -\frac{x^3}{6}.$$

Also, since $\cosh(x) \underset{x\to0}{\longrightarrow} 1$,
$$\ln(\cosh(x)) \underset{x\to0}{\sim} \cosh(x) - 1 \underset{x\to0}{\sim} \frac{x^2}{2}.$$

b) Hence
$$f(x) \underset{x\to0}{\sim} -\frac{x}{3} \underset{x\to0}{\longrightarrow}.$$

Hence f possesses an extension by continuity $\tilde{f}$ at 0 and $\tilde{f}(0) = 0$.

2. We first expand the denominator at order 4. For this, we use the following Taylor–Young expansions:
$$\ln(1+X) \underset{X\to0}{=} X - \frac{X^2}{2} + o(X^2) . X = \cosh(x) - 1 \qquad \underset{x\to0}{=} \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

Hence, since $X \underset{x\to0}{\sim} x^2/2$, we have $o(X^2) \underset{x\to0}{=} o(x^4)$ and hence
$$\ln(\cosh(x)) = \ln(1+X) \underset{X\to0}{=} \left(\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) - \frac{1}{2}\left(\frac{x^2}{2} + o(x^2)\right)^2 + o(x^4)$$
$$\underset{x\to0}{=} \frac{x^2}{2} - \frac{x^4}{12} + o(x^4)$$

We now use the expansion
$$\sin(x) - x \underset{x\to0}{=} -\frac{x^3}{6} + \frac{x^5}{120} + o(x^5),$$

and a long division:
$$\begin{array}{l|l}
 & -\frac{x}{3} - \frac{7x^3}{180} + o(x^3) \\
\hline
\frac{x^2}{2} - \frac{x^4}{12} + o(x^4) & -\frac{x^3}{6} + \frac{x^5}{120} + o(x^5) \\
 & -\left(-\frac{x^3}{6} + \frac{x^5}{36} + o(x^5)\right) \\
\cline{2-2}
 & -\frac{7x^5}{360} + o(x^5) \\
 & -\left(-\frac{7x^5}{360} + o(x^5)\right) \\
\cline{2-2}
 & o(x^5)
\end{array}$$

Hence
$$\tilde{f}(x) \underset{x\to0}{=} -\frac{x}{3} - \frac{7x^3}{180} + o(x^3).$$

Hence $a = 0$, $b = -\frac{1}{3}$, $c = 0$, $d = -\frac{7}{180}$.

3. a) An equation of the tangent line Δ to the graph of $\tilde{f}$ at $(0, \tilde{f}(0))$ is:
$$\Delta : y = -\frac{x}{3}.$$

Moreover, the graph of $\tilde{f}$ is below Δ for $x > 0$ (and x small enough) and the graph of $\tilde{f}$ is above Δ for $x < 0$ (and x small enough).

b) See Figure 3.

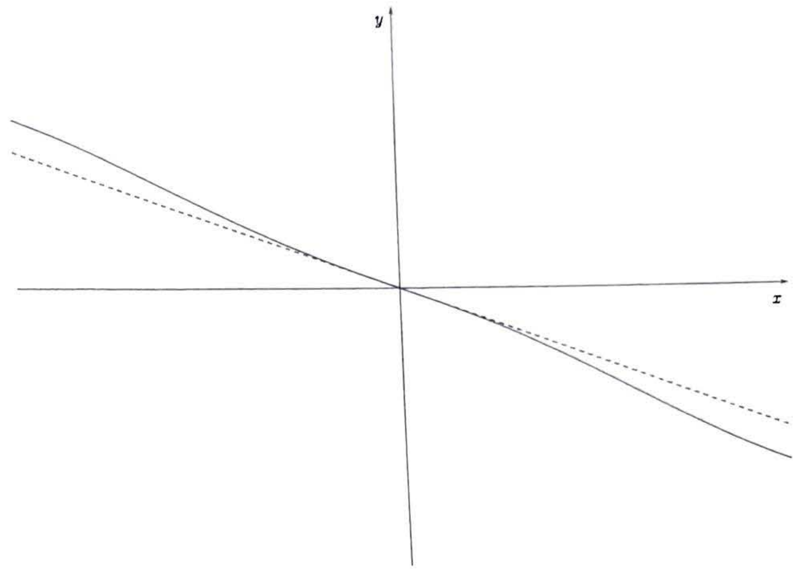

Figure 3 – Graph of $\tilde{f}$ and Δ (in dashed) in a neighborhood of 0

Exercise 4.

1. Let $n \in \mathbb{N}^*$. Since $0 < u_n$ we conclude that $1 < e^{u_n}$, hence, multiplying by u_n (which is positive), we obtain
$$u_n < u_n e^{u_n} = \frac{1}{n}.$$
Hence,
$$0 < u_n < \frac{1}{n},$$
and by the Squeeze Theorem, $u_n \underset{n\to+\infty}{\longrightarrow} 0$.

2. Hence $e^{u_n} \underset{n\to+\infty}{\longrightarrow} 1$, hence $e^{u_n} \underset{n\to+\infty}{\sim} 1$ hence
$$\frac{1}{n} = u_n e^{u_n} \underset{n\to+\infty}{\sim} u_n.$$

3. We know that $e^X \underset{X\to0}{=} 1 + X + o(X)$, hence $Xe^X \underset{X\to0}{=} X + X^2 + o(X^2)$. Now, from the previous question,
$$u_n \underset{n\to+\infty}{=} \frac{1}{n} + o\left(\frac{1}{n}\right).$$
Hence,

- replacing X by u_n (which is valid since $X = u_n \underset{n\to+\infty}{\longrightarrow} 0$),
- using the fact that $o(u_n^2) \underset{n\to+\infty}{=} o\left(\frac{1}{n^2}\right)$ since $u_n \underset{n\to+\infty}{\sim} \frac{1}{n}$

yields
$$\frac{1}{n} = u_n e^{u_n} \underset{n\to+\infty}{=} u_n + u_n^2 + o(u_n^2),$$

hence, from $u_n \underset{n\to+\infty}{=} \frac{1}{n} + o\left(\frac{1}{n}\right)$,

$$\frac{1}{n} = u_n e^{u_n} \underset{n\to+\infty}{=} u_n + \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right)^2 + o\left(\frac{1}{n^2}\right) \underset{n\to+\infty}{=} u_n + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right),$$

and hence

$$u_n \underset{n\to+\infty}{=} \frac{1}{n} - \frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

Exercise 5.

1. For $n \in \mathbb{N}^*$,

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=1}^{n} \frac{1}{1+k/n}\frac{1}{n}$$

where we recognize the Riemann sum associated with the continuous function

$$f : \begin{array}{rcl} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{1+x} \end{array}.$$

and the tagged subdivision $T = \big((x_0,\dots,x_n),(t_1,\dots,t_n)\big)$ of $[0,1]$ given by

$$\forall k \in \{0,\dots,n\},\ x_k = \frac{k}{n}, \qquad \forall k \in \{1,\dots,n\},\ t_k = x_k.$$

Hence

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \int_0^1 \frac{dx}{1+x} = \big[\ln(1+x)\big]_{x=0}^{x=1} = \ln(2).$$

2. a) The sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing since, for $n \in \mathbb{N}^*$,

$$u_{n+1} - u_n = \frac{1}{n+1} > 0.$$

Hence, by the Monotone Limit Theorem, the limit ℓ of $(u_n)_{n\in\mathbb{N}^*}$ exists in $\overline{\mathbb{R}}$. Since $u_1 = 1 > 0$, we must have $\ell \in \mathbb{R}_+^* \cup \{+\infty\}$.

b) Let $n \in \mathbb{N}^*$. Then

$$u_{2n} - u_{n+1} = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{p=1}^{n} \frac{1}{n+p},$$

where we shifted the index using $p = k - n$.

c) We know that $\ell \in \mathbb{R}_+^* \cup \{+\infty\}$. We proceed by contradiction: we assume that $\ell \neq +\infty$. Then, since $\lim_{n\to+\infty} u_{2n} = \ell$ and $\lim_{n\to+\infty} u_{n+1} = \ell$, using the elementary operations on limits yields:

$$\lim_{n\to+\infty} u_{2n} - u_{n+1} = \ell - \ell = 0.$$

But from the preliminary question, we know that this limit is $\ln(2) \neq 0$; we hence obtained a contradiction, and we conclude that $\ell = +\infty$.

Exercise 6.

$$(S)\ \begin{cases} x + y - 2z = 0 \\ 2x - y - z = 0 \\ -x + 2y - z = 0 \end{cases} \underset{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1}}{\iff} \begin{cases} x + y - 2z = 0 \\ -3y + 3z = 0 \\ 3y - 3z = 0 \end{cases} \underset{R_3 \leftarrow R_3 + R_2}{\iff} \begin{cases} x + y - 2z = 0 \\ -3y + 3z = 0 \\ 0 = 0 \end{cases}$$

Hence System (S) is compatible and of rank 2. We choose z as parameter and perform the back-substitution:

$$(S) \iff \begin{cases} x + y - 2z = 0 \\ -3y + 3z = 0 \\ z = z \end{cases} \iff \begin{cases} x = z \\ y = z \\ z = z \end{cases}$$