SCAN 1 — S2 — Math Test #4 — 2h

March 23, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The questions of this exercise are independent from each other.

1. Determine the value of the following limit:

$$\lim_{x\to 0}\left(e^x - x\right)^{1/x^2}.$$

2. Determine the largest subset D of $\mathbb{R}$ such that for all $x \in D$ the following expression

$$\int_x^{x+1} \frac{e^t}{t}\,dt$$

is well-defined.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1, periodic of period 2π. Show, using an integration by parts, that

$$\lim_{n\to+\infty} \int_0^{2\pi} f(t)\sin(nt)\,dt = 0.$$

4. Determine an antiderivative of the function f defined by

$$f : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x+2}{x^2+x+1}. \end{array}$$

5. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \frac{1}{3}.$$

Show that there exists $c \in [0,1]$ such that $f(c) = c^2$.

Exercise 2. The goal of this exercise is to compute a numerical value of $\sin(2/3)$.

1. Show that:

$$\frac{2}{3} - \left(\frac{2}{3}\right)^3\frac{1}{6} + \left(\frac{2}{3}\right)^5\frac{1}{120} - \left(\frac{2}{3}\right)^7\frac{1}{5040} < \sin\left(\frac{2}{3}\right) < \frac{2}{3} - \left(\frac{2}{3}\right)^3\frac{1}{6} + \left(\frac{2}{3}\right)^5\frac{1}{120}.$$

2. With a calculator (one that only computes elementary algebraic operations), we obtain:

$$\frac{2}{3} - \left(\frac{2}{3}\right)^3\frac{1}{6} + \left(\frac{2}{3}\right)^5\frac{1}{120} = \frac{2254}{3645}$$

$$= 0.6\,\overline{183813443072702331961591220850480109739368998628257887517146776406035665294924554}$$

$$\left(\frac{2}{3}\right)^7\frac{1}{5040} = \frac{8}{688905} < 0.000012,$$

(where the line over the digits means to repeat the digits). Deduce an approximation of $\sin(2/3)$ correct to as many decimal places as you can.

Exercise 3. Let f be the function defined by

$$f : \mathbb{R}^* \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{\sin(x) - x}{\ln(\cosh(x))}.$$

1. a) Find the simplest equivalents of

$$\sin(x) - x \qquad \text{and} \qquad \ln(\cosh(x))$$

as $x \to 0$.

b) Deduce that f possesses an extension by continuity at 0, that we denote by $\tilde{f}$. Specify the value of $\tilde{f}(0)$.

2. Show that there exists coefficients $a, b, c, d \in \mathbb{R}$ such that

$$\tilde{f}(x) \underset{x \to 0}{=} a + bx + cx^2 + dx^3 + o(x^3),$$

and determine a, b, c, d explicitly.

3. a) Determine an equation of the tangent line Δ to the graph of $\tilde{f}$ at $(0, \tilde{f}(0))$ and the relative position of the graph of $\tilde{f}$ with respect to Δ.

b) Sketch, on the same figure, the graph of $\tilde{f}$ and of Δ in a neighborhood of 0.

Exercise 4. You are given that for all $n \in \mathbb{N}^*$, there exists a unique element $u_n \in (0,1)$ such that $u_n e^{u_n} = \frac{1}{n}$.

1. Show that $\lim\limits_{n \to +\infty} u_n = 0$.

2. Deduce that $u_n \underset{n \to +\infty}{\sim} \frac{1}{n}$. Do we have $u_n \underset{n \to +\infty}{=} \frac{1}{n} + o\left(\frac{1}{n}\right)$?

3. Deduce that

$$u_n \underset{n \to +\infty}{=} \frac{1}{n} - \frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

Exercise 5.

1. Preliminary question: Show that the following limit exists in $\mathbb{R}$ and determine its value:

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

2. We now define the sequence $(u_n)_{n \in \mathbb{N}}$ by

$$\forall n \in \mathbb{N}^*,\ u_n = \sum_{k=1}^{n} \frac{1}{k}.$$

a) Explain why the limit of the sequence $(u_n)_{n \in \mathbb{N}^*}$ exists in $\mathbb{R}_+^* \cup \{+\infty\}$. We denote by ℓ the value of this limit.

b) Let $n \in \mathbb{N}^*$. Show that

$$u_{2n} - u_{n+1} = \sum_{k=1}^{n} \frac{1}{n+k}.$$

c) Deduce that $\ell = +\infty$.

Exercise 6. Solve (using the Gaussian elimination) the following linear system, and determine its rank:

$$\begin{cases} x + y - 2z = 0 \\ 2x - y - z = 0 \\ -x + 2y - z = 0 \end{cases}$$