INSA — INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Solution of Math Test #5

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Exercise 1.

1. a) Define the function

$$h:\ \mathbb{R}_+^* \longrightarrow \mathbb{R},\qquad t \longmapsto \frac{e^t}{t}.$$

Since h is continuous, h possesses an antiderivative, say H. Now, by the Fundamental Theorem of Calculus,

$$\forall x\in\mathbb{R}_+^*,\ f(x)=H(x+1)-H(x).$$

Since H is differentiable, we conclude (by the chain rule and the usual operations) that f is differentiable and, for all $x\in\mathbb{R}_+^*$:

$$\begin{aligned} f'(x) &= H'(x+1)-H'(x) = h(x+1)-h(x)\\ &= \frac{e^{x+1}}{x+1}-\frac{e^x}{x}\\ &= e^x\left(\frac{e}{x+1}-\frac{1}{x}\right)\\ &= \frac{e^x}{x(x+1)}\big((e-1)x-1\big). \end{aligned}$$

b) From the previous form of f', we conclude that

$$\forall x\in\mathbb{R}_+^*,\ f'(x)>0 \iff x>\frac{1}{e-1} \text{ and } f'(x)<0 \iff x<\frac{1}{e-1},$$

hence f is increasing on $\big[1/(e-1),+\infty\big)$ and f is decreasing on $\big(0,1/(e-1)\big]$.

2. a) Let $x\in\mathbb{R}_+^*$. The function exp is continuous on $[x,x+1]$ and the function $t\mapsto 1/t$ is (piecewise) continuous and positive on $[x,x+1]$ hence, by the Mean Value Theorem (MVT2), there exists $c\in[x,x+1]$ such that

$$f(x)=e^c\int_x^{x+1}\frac{dt}{t}=e^c\big(\ln(x+1)-\ln(x)\big).$$

b) Hence (since the c in the previous statement is between x and $x+1$, and exp is increasing), we have

$$\forall x\in\mathbb{R}_+^*,\ e^x\big(\ln(x+1)-\ln(x)\big)\le f(x).$$

Now, by the elementary operations on limits,

$$\lim_{x\to 0^+} e^x\big(\ln(x+1)-\ln(x)\big)=1\times\big(0-(-\infty)\big)=+\infty,$$

and we conclude, by the Squeeze Theorem, that $\lim\limits_{x\to0^+}f(x)=+\infty$.

3. a) Let $x\in\mathbb{R}_+^*$. The function exp is continuous and the function $t\mapsto 1/t$ is of class C^1 hence

$$\begin{aligned} f(x) &= \int_x^{x+1}\frac{e^t}{t}\,dt\\ &= \left[\frac{e^t}{t}\right]_{t=x}^{t=x+1}-\int_x^{x+1}\frac{e^t}{-t^2}\,dt\\ &= \frac{e^{x+1}}{x+1}-\frac{e^x}{x}+\int_x^{x+1}\frac{e^t}{t^2}\,dt\\ &= e^x\left(\frac{e}{x+1}-\frac{1}{x}\right)+\int_x^{x+1}\frac{e^t}{t^2}\,dt. \end{aligned}$$

b) Let $x\in\mathbb{R}_+^*$. The function $t\mapsto 1/t$ is continuous on $[x,x+1]$ and the function exp is (piecewise) continuous and positive on $[x,x+1]$ hence, by the Mean Value Theorem (MVT2), there exists $c_x\in[x,x+1]$ such that

$$\int_x^{x+1}\frac{e^t}{t^2}\,dt=\frac{1}{c_x^2}\int_x^{x+1}e^t\,dt=\frac{1}{c_x^2}\left(e^{x+1}-e^x\right)=\frac{1}{c_x^2}e^x(e-1).$$

Now since $c_x\in[x,x+1]$,

$$\frac{1}{(x+1)^2}\le\frac{1}{c_x^2}\le\frac{1}{x^2},$$

hence

$$\frac{x^2}{(x+1)^2}\le\frac{x^2}{c_x^2}\le 1,$$

and by the Squeeze Theorem,

$$\frac{x^2}{c_x^2}\xrightarrow[x\to+\infty]{}1,$$

hence $c_x \underset{x\to+\infty}{\sim} 1$. We hence conclude that

$$\int_x^{x+1}\frac{e^t}{t^2}\,dt\underset{x\to+\infty}{\sim}\frac{e^x}{x^2}(e-1).$$

Yes,

$$\int_x^{x+1}\frac{e^t}{t^2}\,dt\underset{x\to+\infty}{=}e^x\left(\frac{e-1}{x^2}+o\left(\frac{1}{x^2}\right)\right).$$

c) For $x\in\mathbb{R}_+^*$,

$$f(x)=e^x\left(\frac{e}{x+1}-\frac{1}{x}\right)+\int_x^{x+1}\frac{e^t}{t^2}\,dt.$$

Now,

$$\frac{1}{x+1}=\frac{1}{x}\times\frac{1}{1+\frac{1}{x}}\underset{x\to+\infty}{=}\frac{1}{x}\left(1-\frac{1}{x}+o\left(\frac{1}{x}\right)\right)\underset{x\to+\infty}{=}\frac{1}{x}-\frac{1}{x^2}+o\left(\frac{1}{x^2}\right).$$

Hence,

$$\begin{aligned} f(x) &\underset{x\to+\infty}{=} e^x\left(e\left(\frac{1}{x}-\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)-\frac{1}{x}\right)+e^x\left(\frac{e-1}{x^2}+o\left(\frac{1}{x^2}\right)\right)\\ &\underset{x\to+\infty}{=} e^x\left(\frac{e-1}{x}-\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right). \end{aligned}$$

Exercise 2.

1. With the substitution $u=\sqrt{\cos(x)}$:

- $du=-\dfrac{\sin(x)}{2\sqrt{\cos(x)}}\,dx$,
- when $x=0$, $u=1$,
- when $x=\pi/3$, $u=1/\sqrt{2}$.

Then:

$$\begin{aligned} I &= -2\int_0^{\pi/3}\sin^2(x)\left(-\frac{\sin(x)}{2\sqrt{\cos(x)}}\right)dx\\ &= -2\int_0^{\pi/3}\left(1-\cos^2(x)\right)\left(-\frac{\sin(x)}{2\sqrt{\cos(x)}}\right)dx\\ &= -2\int_1^{1/\sqrt{2}}\left(1-u^4\right)du\\ &= -2\left[u-\frac{u^5}{5}\right]_{u=1}^{u=1/\sqrt{2}} \end{aligned}$$

$$= -2\left(\frac{1}{\sqrt{2}} - \frac{1}{20\sqrt{2}} - 1 + \frac{1}{5}\right)$$
$$= -\frac{19\sqrt{2}}{20} + \frac{8}{5}$$

2. a)

$$J = \int_0^1 e^{x\ln(2)}\,dx = \left[\frac{e^{x\ln(2)}}{\ln(2)}\right]_{x=0}^{x=1} = \frac{1}{\ln(2)}.$$

b) Let $n \in \mathbb{N}^*$ and consider the tagged subdivision $T_n = ((x_0,\dots,x_n),(t_1,\dots,t_n))$ of $[0,1]$ where:

$$\forall k \in \{0,\dots,n\},\ x_k = \frac{k}{n},\ \forall k \in \{1,\dots,n\},\ t_k = x_k.$$

Then the Riemann sum of (the continuous function on $[0,1]$) $x \mapsto 2^x$ associated with T_n is:

$$R_n = \sum_{k=1}^{n} 2^{t_k}(x_k - x_{k-1}) = \sum_{k=1}^{n} 2^{k/n}\frac{1}{n} = \sum_{k=1}^{n} \frac{\sqrt[n]{2^k}}{n}.$$

Hence, $\ell = J = 1/\ln(2)$.

3. We notice that the given denominator is already fully factored in $\mathbb{R}$ and that the degree of the numerator is less than that of the denominator. Hence the general form of the decomposition is:

$$F(x) = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

- To determine A: multiply by $x-1$ and evaluate at $x=1$: $-3 = A$.
- To determine B we can multiply by x and take the limit $x \to +\infty$: $1 = A + B$ hence $B = 4$.
- To determine C we now can evaluate, e.g., at $x = 0$: $4 = -A + C/4$ hence $C = 4$.

Conclusion:

$$F(x) = -\frac{3}{x-1} + \frac{4x+4}{x^2+4}.$$

Exercise 3.

1. $\mathscr{B} = (1, X, X^2, X^3)$. $\dim E = 4$.

2.
   - Clearly the nil vector 0_E belongs to F, hence $F \neq \emptyset$.
   - Let $P, Q \in F$ and let $\lambda, \mu \in \mathbb{R}$. Then

$$(\lambda P+\mu Q)(0)+3(\lambda P+\mu Q)(1) = \lambda P(0)+\mu Q(0)+3\lambda P(1)+3\mu Q(1) = \lambda\big(P(0)+3P(1)\big)+\mu\big(Q(0)+3Q(1)\big) = 0.$$

   Hence $\lambda P + \mu Q \in F$.

   Hence F is a subspace of E.

3. Let $P, Q \in E$ and let $\lambda \in \mathbb{R}$. Then

$$f(P+\lambda Q) = (P+\lambda Q)(0) + 3(P+\lambda Q)(1) = P(0) + 3P(1) + \lambda\big(Q(0)+3Q(1)\big) = f(P) + \lambda f(Q).$$

Hence f is linear.

The relation between f and F is: $F = \operatorname{Ker} f$.

Since f is a non-nil linear *form*, f is onto, hence $\operatorname{rk} f = 1$. By the Rank–Nullity Theorem, we conclude that $\dim \operatorname{Ker} f = 4 - 1 = 3$ hence $\dim F = 3$.

Exercise 4.

1.

$$\dim F = \operatorname{rk}(a,b,c) = \operatorname{rk}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ -1 & 2 & -5 \\ -1 & 2 & -5 \end{pmatrix} \underset{\substack{C_2 \leftarrow C_2 - C_1 \\ C_3 \leftarrow C_3 - 2C_1}}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 3 & -3 \\ -1 & 3 & -3 \end{pmatrix} = 2.$$

Since a and b are two non-collinear vectors, the family (a,b) is an independent family (of vectors of F). Since $\dim F = 2$, we conclude that the family (a,b) is also a generating family of F, hence (a,b) is a basis of F.

Since the vectors d and e are non-collinear, the family (d,e) is an independent family (of vectors of G). By definition, the family (d,e) is also a generating family of G, hence (d,e) is a basis of G. Hence $\dim G = 2$.

2. We know that $F + G = \operatorname{Span}\{a,b,d,e\}$. Now,

$$\dim(F+G) = \operatorname{rk}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & -4 & 2 \\ -1 & 2 & 3 & 1 \\ -1 & 2 & 1 & 1 \end{pmatrix}$$
$$\underset{\substack{C_2 \leftarrow C_2 - C_1 \\ C_4 \leftarrow C_4 + C_1}}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -4 & 3 \\ -1 & 3 & 3 & 0 \\ -1 & 3 & 3 & 0 \end{pmatrix}$$
$$\underset{\substack{C_3 \leftarrow C_3 + 4C_1 \\ C_4 \leftarrow C_4 - 3C_1}}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 3 & 15 & -9 \\ -1 & 3 & 15 & -9 \end{pmatrix} = 3.$$

3. By Grassmann's Formula,

$$\dim(F \cap G) = \dim F + \dim G - \dim(F+G) = 2 + 2 - 3 = 1.$$

Hence $F \cap G \neq \{0_E\}$, hence the subspaces F and G are not independent, hence the sum $F + G$ is not a direct sum.

Exercise 5.

1. Let $m \in \mathbb{R}$.

   a) Let $a, b, c \in \mathbb{R}$.

$$(S) \underset{R_1 \leftrightarrow R_2}{\iff} \begin{cases} x + my + z = b \\ mx + y + z = a \\ x + y + mz = c \end{cases} \underset{\substack{R_2 \leftarrow R_2 - mR_1 \\ R_3 \leftarrow R_3 - R_1}}{\iff} \begin{cases} x + my + z = b \\ (1-m^2)y + (1-m)z = a - mb \\ (1-m)y + (m-1)z = c - b \end{cases}$$

$$\iff \begin{cases} x + z + my = b \\ (1-m)z + (1-m^2)y = a - mb \\ (m-1)z + (1-m)y = c - b \end{cases}$$

$$\underset{R_3 \leftarrow R_3 + R_2}{\iff} \begin{cases} x + z + my = b \\ (1-m)z + (1-m^2)y = a - mb \\ (2-m-m^2)y = a - (m+1)b + c \end{cases}$$

Notice that $2 - m - m^2 = -(m-1)(m+2)$, so we consider the cases $m = 1$ and $m = -2$ separately:

- if $m = -2$:

$$(S) \iff \begin{cases} x + z - 2y = b \\ 3z - 3y = a + 2b \\ 0 = a + b + c, \end{cases}$$

  and the rank of (S_{-2}) is 2, and (S_{-2}) possesses solutions if and only if $a + b + c = 0$.

- If $m = 1$:

$$(S) \iff \begin{cases} x + z + y = b \\ 0 = a - b \\ 0 = a - 2b + c. \end{cases}$$

  The rank of (S_1) is 1, and (S_1) possesses solutions if and only if $a = b = c$.

- If $m \notin \{-2, 1\}$, the rank of (S_m) is 3, and (S_m) possesses a unique solution.

b) We check for independence of the family $\mathscr{B}$: let $x, y, z \in \mathbb{R}$. Then

$$xu + yv + zw = 0_E \iff \begin{cases} mx + y + z = 0 \\ x + my + z = 0 \\ x + y + mz = 0. \end{cases}$$

From the previous question, we know that the system possesses a unique solution if and only if $m \notin \{-2, 1\}$, hence $\mathscr{B}$ is independent if and only if $m \notin \{-2, 1\}$. Now since $\dim E = 3 = \#\mathscr{B}$, we conclude that $\mathscr{B}$ is a basis of E if and only if $m \notin \{-2, 1\}$.

2.

$$(S_2) \iff \begin{cases} x + z + 2y = b \\ - z - 3y = a - 2b \\ - 4y = a - 3b + c \end{cases} \iff \begin{cases} x = \frac{3}{4}a - \frac{1}{4}b - \frac{1}{4}c \\ z = -\frac{1}{4}a - \frac{1}{4}b + \frac{3}{4}c \\ y = -\frac{1}{4}a + \frac{3}{4}b - \frac{1}{4}c \end{cases}$$

3. We know that *a linear map is uniquely determined by the image of a basis of its domain*. Here we know that (u, v, w) is a basis of E, hence there exists a unique such linear map.

4. We know that a generating family of $\operatorname{Im} f$ is given by $f(\mathscr{B}) = \big(f(u), f(v), f(w)\big)$. Now,

$$\operatorname{Im} f = \operatorname{rk} f(\mathscr{B}) = \operatorname{rk}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \underset{\substack{C_2 \leftarrow C_2 - C_1 \\ C_3 \leftarrow C_3 - 2C_1}}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix} = 2.$$

Hence $\operatorname{rk} f = 2$. By the Rank–Nullity Theorem, $\dim \operatorname{Ker} f = 3 - 2 = 1$.

5.

$$[f]_{\mathscr{B},\text{std}} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

6. From the solutions of the system (S_2), we know that

$$(1,0,0) = \frac{3}{4}u - \frac{1}{4}v - \frac{1}{4}w,$$

hence

$$f(1,0,0) = (0, -1/2, -1),$$

and

$$(0,1,0) = -\frac{1}{4}u + \frac{3}{4}v - \frac{1}{4}w,$$

hence

$$f(0,1,0) = (0, 1/2, 1),$$

and

$$(0,0,1) = -\frac{1}{4}u - \frac{1}{4}v + \frac{3}{4}w,$$

hence

$$f(0,0,1) = (1, 1/2, 0).$$

Hence

$$[f]_{\text{std}} = \begin{pmatrix} 0 & 0 & 1 \\ -1/2 & 1/2 & 1/2 \\ -1 & 1 & 0 \end{pmatrix}.$$

Exercise 6. Let $\lambda, \mu, \nu \in \mathbb{R}$ such that $\lambda u + \mu v + \nu w$, i.e.,

$$\forall x \in \mathbb{R},\ \lambda \sin(x) + \mu \cos(x) + \nu e^x = 0.$$

If $\nu \neq 0$, then (since sin and cos are bounded),

$$\lim_{x \to +\infty} \lambda \sin(x) + \mu \cos(x) + \nu e^x \begin{cases} +\infty & \text{if } \nu > 0 \\ -\infty & \text{if } \nu < 0 \end{cases}$$

but this is impossible since this limit must be nil. Hence $\nu = 0$. Now, evaluating at $x = 0$ yields $\mu = 0$, and evaluating at $\pi/2$ yields $\lambda = 0$. Hence we must have $\lambda = \mu = \nu = 0$, hence the family (u, v, w) is independent.

Exercise 7.

1.
- Since f is a linear map, we must have $f(0_E) = 0_F$, hence $0_E \in \operatorname{Ker} f$, hence $\operatorname{Ker} f \neq \emptyset$.
- Let $u, v \in \operatorname{Ker} f$ and let $\lambda, \mu \in \mathbb{K}$. Then

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) = 0_F$$

hence $\lambda u + \mu v \in \operatorname{Ker} f$.

2.
- Assume that f is injective, and let's show that $\operatorname{Ker} f = \{0_E\}$: Let $u \in \operatorname{Ker} f$. Then $f(u) = 0_F$. But since f is linear, we know that $f(0_E) = 0_F$, hence $f(u) = f(0_E)$. Since f is injective, we must have $u = 0_E$. Hence $\operatorname{Ker} f = \{0_E\}$.
- Assume that $\operatorname{Ker} f = \{0_E\}$ and let's show that f is injective: let $u, v \in E$ such that $f(u) - f(v)$. Then, since f is linear, $f(u - v) = 0_F$ that is, $u - v \in \operatorname{Ker} f$. Since $\operatorname{Ker} f = \{0_E\}$, we must have $u - v = 0_E$, hence $u = v$. Hence f is injective.