INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S2 — Math Test #5 — 2h

May 18, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. We define the function

$$f : \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\int_x^{x+1} \frac{e^t}{t}\,dt. \end{array}$$

1. a) Show that f is differentiable and determine an explicit expression of f'.
   b) Determine the variations of f on $\mathbb{R}_+^*$.
2. a) Let $x \in \mathbb{R}_+^*$. Show that there exists $c \in [x, x+1]$ such that
   $$f(x) = e^c\big(\ln(x+1) - \ln(x)\big).$$
   b) Deduce the value of $\lim\limits_{x \to 0^+} f(x)$.
3. a) Use an integration by parts to show that
   $$\forall x \in \mathbb{R}_+^*,\ f(x) = e^x\left(\frac{e}{x+1} - \frac{1}{x}\right) + \int_x^{x+1} \frac{e^t}{t^2}\,dt.$$
   b) Show, using the Mean Value Theorem (MVT2), that
   $$\int_x^{x+1} \frac{e^t}{t^2}\,dt \underset{x \to +\infty}{\sim} (e-1)\frac{e^x}{x^2}.$$
   Is the following property true?
   $$\int_x^{x+1} \frac{e^t}{t^2}\,dt \underset{x \to +\infty}{=} e^x\left(\frac{e-1}{x^2} + o\left(\frac{1}{x^2}\right)\right).$$
   c) Deduce that
   $$f(x) \underset{x \to +\infty}{=} e^x\left(\frac{e-1}{x} - \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right).$$

$\cos \frac{\pi}{3}$

Exercise 2. The three questions of this exercise are independent from each other.

1. Use the substitution $u = \sqrt{\cos(x)}$ to show that
   $$I = \int_0^{\pi/3} \frac{\sin^3(x)}{\sqrt{\cos(x)}}\,dx = \frac{8}{5} - \frac{19}{20}\sqrt{2}.$$

$\left(\frac{1}{2}\right)^4$

2. a) Compute the value of the following integral:
   $$J = \int_0^1 2^x\,dx.$$
   b) Deduce the value of the following limit:
   $$\ell = \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{\sqrt[n]{2^k}}{n}.$$
3. Give the partial fraction decomposition of the following rational fraction:
   $$F(x) = \frac{x^2 - 16}{(x-1)(x^2+4)}.$$

Exercise 3. Let $E = \mathbb{R}_3[X]$ be the real vector space of formal polynomials with real coefficients, indeterminate X, and of degree non-greater than 3.

1. Recall (without any justifications) the standard basis $\mathscr{B}$ of E, as well as the dimension of E.
2. Let
$$F = \{P \in E \mid P(0) + 3P(1) = 0\}.$$
Show that F is a subspace of E.
3. Define the mapping
$$\begin{aligned} f : E &\longrightarrow \mathbb{R} \\ P &\longmapsto P(0) + 3P(1). \end{aligned}$$
Show that f is linear. How are f and F related? What is the rank of f? Can you deduce the dimension of F?

Exercise 4. We define the following vectors of $E = \mathbb{R}^4$:

$$a = (1, 1, -1, -1), \qquad b = (1, 2, 2, 2), \qquad c = (2, 1, -5, -5), \qquad d = (0, -4, 3, 3), \qquad e = (-1, 2, 1, 1),$$

and we define

$$F = \operatorname{Span}\{a, b, c\}, \qquad G = \operatorname{Span}\{d, e\}.$$

1. Determine the dimension and a basis of F and G.
2. Determine the dimension of $F + G$.
3. Deduce the dimension of $F \cap G$. Is the sum $F + G$ a direct sum?

Exercise 5. Let $E = \mathbb{R}^3$.

1. Let $m \in \mathbb{R}$.
   a) Let $a, b, c \in \mathbb{R}$. Give the rank of the following linear system (in terms of m), **but don't solve it!**
   $$(S_m) \qquad \begin{cases} mx + y + z = a \\ x + my + z = b \\ x + y + mz = c. \end{cases}$$
   Give conditions (on a, b and c, depending on m) under which the system (S_m) possesses solutions.
   b) We define the following vectors of E:
   $$u = (m, 1, 1), \qquad v = (1, m, 1), \qquad w = (1, 1, m).$$
   For what value(s) of m is the family $\mathscr{B} = (u, v, w)$ a basis of E?
2. From now on we consider the case $m = 2$. From the previous question, we know that $\mathscr{B}$ is a basis of E.
   a) Give the solutions of System (S_2) (that is, give the solutions of the system of question 1, in the special case $m = 2$).
   b) Explain why there exists a unique endomorphism $f : E \longrightarrow E$ such that
   $$f(u) = (1, 0, -1), \qquad f(v) = (1, 1, 1), \qquad f(w) = (2, 1, 0).$$
   c) Determine the rank of f and deduce the dimension of $\operatorname{Ker} f$.
   d) Give the matrix $[f]_{\mathscr{B},\mathrm{std}}$ of f in the bases $\mathscr{B}$ and the standard basis std of E.
   e) Give the matrix $[f]_{\mathrm{std}}$ of f in the standard basis.

Exercise 6. Let $E = \mathbb{R}^{\mathbb{R}}$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. We define the following vectors of E:

$$u = \sin, \qquad v = \cos, \qquad w = \exp.$$

Is the family (u, v, w) an independent family of E? justify your answer.

Exercise 7. Let E and F be two vector spaces over $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), and let $f : E \longrightarrow F$ be a linear map.

1. Show that $\operatorname{Ker} f$ is a subspace of E.
2. Show that f is injective if and only if $\operatorname{Ker} f = \{0_E\}$.

$\lambda^2 \mp 4\lambda + 5$