SCAN 1 — S2 — Math Test #6 — 2h

June 12, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. We define the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & -1 & 0 \end{pmatrix},$$

and the endomorphism f of $\mathbb{R}^3$ such that $[f]_{\mathrm{std}} = A$.

1. a) Without computing the characteristic polynomial of A, show that 1 and 2 are eigenvalues of A.
   b) Deduce all the eigenvalues of A and their multiplicities.
   c) Determine the dimension of the eigenspaces of A.
   d) Is A diagonalizable? justify your answer (as concisely as you can).
2. We define the following matrix of $\mathbb{R}^3$:

$$P = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{pmatrix}.$$

   Show that the matrix P is invertible and compute P^{-1}.
3. We define the following vectors of $\mathbb{R}^3$:

$$u_1 = (1, 1, -1), \qquad u_2 = (-2, -1, 2), \qquad u_3 = (0, -1, 1).$$

   a) Briefly explain why $\mathscr{B} = (u_1, u_2, u_3)$ is a basis of $\mathbb{R}^3$.
   b) Show that the matrix of f in the basis $\mathscr{B}$ is

$$T = [f]_{\mathscr{B}} = \begin{pmatrix} 2 & \alpha & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   where α is a real number you will determine. What relation exists between the matrices A, P and T?
4. We set

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   and $N = T - D$.
   a) For $k \in \mathbb{N}$, give a general formula for N^k.
   b) For $n \in \mathbb{N}$, give a general formula for T^n.
   c) For $n \in \mathbb{N}$, give a general formula for A^n.

Exercise 2. Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We assume that P is invertible, that is, $ad - bc \neq 0$.
Check that

$$P^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Application: for $\theta \in \mathbb{R}$, we recall that the rotation matrix of angle θ is

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that R_θ is invertible and that $R_\theta^{-1} = R_{-\theta}$.

Exercise 3. Let $E = \mathbb{R}^{\mathbb{N}}$ be the vector space of all real sequences indexed by $\mathbb{N}$ and let

$$F = \left\{ (u_n)_{n\in\mathbb{N}} \in E \,\middle|\, \forall n \in \mathbb{N},\ u_{n+2} - 3u_{n+1} + 2u_n = 0 \right\}.$$

You're given that F is a subspace of E and you don't need to justify this fact.
Let $(u_n)_{n\in\mathbb{N}} \in E$ and define:

$$\forall n \in \mathbb{N},\ X_n = \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix}.$$

1. Find a matrix $A \in M_2(\mathbb{R})$ such that

$$(u_n)_{n\in\mathbb{N}} \in F \iff \forall n \in \mathbb{N},\ X_{n+1} = AX_n.$$

2. Show that A is diagonalizable, and find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.
3. For $n \in \mathbb{N}$, give a general formula for A^n. You may want to use the result of Exercise 2 to efficiently compute P^{-1}.
4. Show that the proposition

(P) $$\forall n \in \mathbb{N},\ X_{n+1} = AX_n$$

is equivalent to the proposition

(Q) $$\forall n \in \mathbb{N},\ X_n = A^n X_0.$$

5. Deduce the general form of the elements of F.

Exercise 4. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

1. Determine the eigenvalues of A, their multiplicities, and deduce that A is diagonalizable.
2. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
3. We now consider the following differential system:

(S) $$\begin{cases} x'(t) = 2x(t) + y(t) \\ y'(t) = x(t) + 2y(t). \end{cases}$$

Let x and y be two real functions on $\mathbb{R}$ of class C^2. For $t \in \mathbb{R}$ define $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $Y(t) = P^{-1}X(t)$. We denote by u and v the components of Y, that is,

$$\forall t \in \mathbb{R},\ Y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

You're given that

$$\forall t \in \mathbb{R},\ Y'(t) = \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = P^{-1} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

(which is pretty much obvious since P is constant).

a) Quickly check that x and y are solutions of System (S) if and only if:

(∗) $$\forall t \in \mathbb{R},\ X'(t) = AX(t).$$

b) Deduce that x and y are solutions of System (S) if and only if:

(∗∗) $$\forall t \in \mathbb{R},\ Y'(t) = DY(t).$$

c) Give the general solution Y of (∗∗) and deduce the general solution x and y of System (S).

Exercise 5. Let f be an endomorphism of $\mathbb{R}^2$ such that f is diagonalizable, f has two eigenvalues -1 and 2 and the associated eigenspaces E_{-1} and E_2 are as in Figure 1 A vector u is also shown on Figure 1. Plot $f(u)$ on Figure 1. Don't forget to hand in the sheet with your paper.

Name:

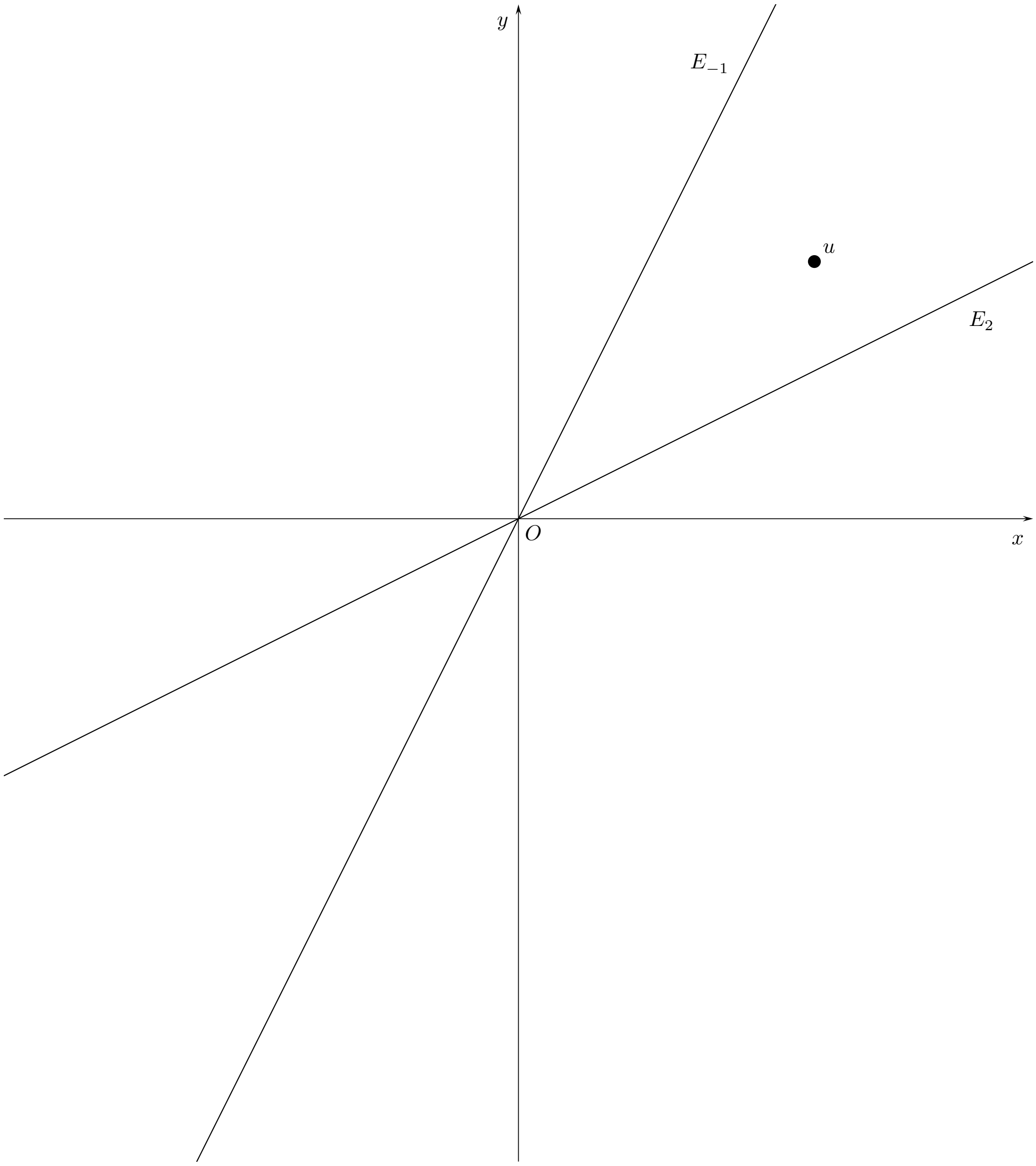

Figure 1 – Plot the image by f of u. The subspaces E_{-1} and E_2 are the eigenspaces of f associated with the eigenvalues -1 and 2 respectively.

This sheet must be handed in with your test!