INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Solution of Math Test #2

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Exercise 1 (*Differential Equations*).

1. Let $y_0, v_0 \in \mathbb{R}$.

   a) We first find the general solution of the following differential equation

$$(*) \qquad y'' - y' - 2y = 0.$$

Its characteristic equation is

$$r^2 - r - 2 = 0.$$

Its discriminant is $\Delta = 1 + 8 = 9 > 0$, and we have two real roots $r_1 = -1$ and $r_2 = 2$, so the general solution of $(*)$ is

$$y(x) = A\mathrm{e}^{2x} + B\mathrm{e}^{-x},$$

where $A, B \in \mathbb{R}$ are constants we're now going to determine using the initial conditions: The first initial condition yields

$$y(0) = A + B = y_0.$$

For the second initial condition, we need to differentiate y:

$$y'(x) = 2A\mathrm{e}^{2x} - B\mathrm{e}^{-x}$$

and we obtain

$$y'(0) = 2A - B = v_0.$$

We hence need to solve

$$\begin{cases} A + B = y_0 \\ 2A - B = v_0. \end{cases}$$

If we add the two rows we obtain $3A = y_0 + v_0$ hence $A = (y_0 + v_0)/3$ and we then obtain $B = y_0 - A = (2y_0 - v_0)/3$. Hence the solution of (IVP) is

$$y(x) = \frac{y_0 + v_0}{3}\mathrm{e}^{2x} + \frac{2y_0 - v_0}{3}\mathrm{e}^{-x}.$$

   b)
   - If $y_0 = -v_0$, then our solution is:

$$y(x) = y_0\mathrm{e}^{-x} \xrightarrow[x\to+\infty]{} 0,$$

   - If $y_0 \neq -v_0$, then

$$\begin{aligned} y(x) &= \frac{y_0 + v_0}{3}\mathrm{e}^{2x} + \frac{2y_0 - v_0}{3}\mathrm{e}^{-x} \\ &= \mathrm{e}^{2x}\left(\frac{y_0 + v_0}{2} + \frac{2y_0 - v_0}{3}\mathrm{e}^{-3x}\right). \end{aligned}$$

     Now,

$$\lim_{x\to+\infty} \frac{y_0 + v_0}{2} + \frac{2y_0 - v_0}{3}\mathrm{e}^{-3x} = \frac{y_0 + v_0}{3} \neq 0$$

     hence, since $\lim_{x\to+\infty} \mathrm{e}^{3x} = +\infty$, we conclude, by the elementary operations on limits, that

$$\begin{aligned} \lim_{x\to+\infty} y(x) &= \begin{cases} +\infty & \text{if } y_0 + v_0 > 0 \\ -\infty & \text{if } y_0 + v_0 < 0 \end{cases} \\ &\neq 0. \end{aligned}$$

2.
   - We need the general solution of the associated homogeneous equation, i.e., of the following differential equation:

$$y' - 3y = 0,$$

     which has the following general solution:

$$y(x) = A\mathrm{e}^{3x},\ A \in \mathbb{R}.$$

- We need a particular solution of $(*)$. To do that, we first consider the complex equation

  $$y'(x) - 3y(x) = 2\mathrm{e}^{3ix}.$$

  We now search for a particular solution of this complex equation of the form $y_{\mathbb{C}}(x) = C\mathrm{e}^{3ix}$. Since

  $$y_{\mathbb{C}}'(x) = 3iC\mathrm{e}^{3ix},$$

  we conclude that:

  $$\begin{aligned} y_{\mathbb{C}} \text{ is a solution of the complex equation} &\iff \forall x \in \mathbb{R},\ 3iC\mathrm{e}^{3ix} - 3C\mathrm{e}^{3ix} = 2\mathrm{e}^{3ix} \\ &\iff (-3+3i)C = 2 \\ &\iff C = \frac{2}{-3+3i}. \end{aligned}$$

  Now we know that a particular solution of $(*)$ is given by

  $$y_p(x) = \operatorname{Re}\bigl(C\mathrm{e}^{3ix}\bigr) = \operatorname{Re}\left(2\frac{\cos(3x) + i\sin(3x)}{-3+3i}\right) = 2\frac{-3\cos(3x) + 3\sin(3x)}{18} = -\frac{1}{3}\cos(3x) + \frac{1}{3}\sin(3x).$$

Finally, we conclude that the general solution of $(*)$ is:

$$y(x) = A\mathrm{e}^{3x} - \frac{1}{3}\cos(3x) - \frac{1}{3}\sin(3x).$$

3. From the part

   $$\mathrm{e}^{-3x}\bigl(A\cos(2x) + B\sin(2x)\bigr),$$

   we conclude that the roots of the characteristic equation of the associated homogeneous equation must be $-3+2i$ and $-3-2i$. Now a polynomial (in r) of degree 2 with these roots is:

   $$(r+3-2i)(r+3+2i) = r^2 + 6r + 13.$$

   Hence the associated homogeneous equation must be

   $$y'' + 6y' + 13y = 0.$$

   Because a particular solution of the differential equation we're looking for is 1, the differential equation must be:

   $$y'' + 6y' + 13y = 13.$$

Exercise 2.

1. $$\forall x, y \in \mathbb{R},\ \sinh(x+y) = \sinh(x)\cosh(y) + \cosh(y)\sinh(x).$$

2. If we had an $\alpha \in \mathbb{R}$ such that $\sinh(\alpha) = 4$ and $\cosh(\alpha) = 5$ we would have, by the Pythagorean Theorem, $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$ i.e., $5^2 - 4^2 = 1$, i.e., $25 - 16 = 1$, i.e., $9 = 1$, which is impossible.

3. Let $x \in \mathbb{R}$. If we divide Equation $(*)$ by a number $\mu \in \mathbb{R}_+^*$, we obtain:

   $$(*) \iff \frac{4}{\mu}\cosh(x) + \frac{5}{\mu}\sinh(x) = \frac{6}{\mu}$$

   We need to find μ so that the system

   $$\begin{cases} \sinh(\alpha) = 4/\mu \\ \cosh(\alpha) = 5/\mu, \end{cases}$$

   and the Pythagorean Theorem forces:

   $$1 = \cosh^2(\alpha) - \sinh^2(\alpha) = \frac{25}{\mu^2} - \frac{16}{\mu^2} = \frac{9}{\mu^2},$$

   and hence (since we want $\mu > 0$), $\mu = 3$. In this case, we can take $\alpha = \operatorname{arcsinh}(4/3)$.

4. Finally, for $x \in \mathbb{R}$,

$$\begin{aligned} x \text{ is a solution of } (*) &\iff \sinh(\alpha + x) = 2 \\ &\iff \alpha + x = \operatorname{arcsinh}(2) \\ &\iff x = \operatorname{arcsinh}(2) - \alpha = \operatorname{arcsinh}(2) - \operatorname{arcsinh}(4/3) \\ &\iff x = \ln\big(2+\sqrt{5}\big) - \ln(3) = \ln\left(\frac{2+\sqrt{5}}{3}\right). \end{aligned}$$

Exercise 3.

1. We notice that -1 and 2 are common roots of the denominators, and their factored form is:

$$x^3 - 3x - 2 = (x-2)(x+1)^2 \qquad \text{and} \qquad x^2 - x - 2 = (x-2)(x+1).$$

Hence, for $x \in (-1,2) \cup (2,+\infty)$ (which is a punctured neighborhood of 2),

$$\begin{aligned} \frac{3}{x^3-3x-2} - \frac{1}{x^2-x-2} &= \frac{3}{(x-2)(x+1)^2} - \frac{1}{(x-2)(x+1)} \\ &= \frac{3-x-1}{(x-2)(x+1)^2} \\ &= \frac{2-x}{(x-2)(x+1)^2} \\ &= -\frac{1}{(x+1)^2} \\ &\underset{x\to 2}{\longrightarrow} -\frac{1}{9}. \end{aligned}$$

2. First observe that, for $x \in \mathbb{R}_+^*$ (which is a neighborhood of $+\infty$) one has:

$$\frac{x\sqrt{2+x^2}}{1+x^3} = \frac{x^2\sqrt{1+2/x^2}}{x^3\big(1+1/x^3\big)} = \frac{1}{x}\,\frac{\sqrt{1+2/x^2}}{\big(1+1/x^3\big)} \underset{x\to+\infty}{\longrightarrow} 0 \times \frac{1}{1} = 0.$$

Now, if $f : \mathbb{R} \to [1,+\infty)$, we have:

$$\forall x \in \mathbb{R},\ 0 < \frac{1}{f(x)} \leq 1.$$

In particular, $1/f$ is bounded. Now we recognized that the expression

$$\frac{x\sqrt{2+x^2}}{\big(1+x^3\big)f(x)} = \frac{x\sqrt{2+x^2}}{\big(1+x^3\big)}\,\frac{1}{f(x)}$$

is the product of a term that goes to 0 by a bounded term. Hence, by the Squeeze Theorem,

$$\lim_{x\to+\infty} \frac{x\sqrt{2+x^2}}{\big(1+x^3\big)f(x)} = 0.$$

3. a) Since

$$\lim_{x\to 0} 1 + x^2 = \lim_{x\to 0} 1 + 2x^2 = 1$$

we conclude, by the Squeeze Theorem, that

$$\lim_{x\to 0} f(x) = 1.$$

Hence $a = 1$.

b) Since

$$\lim_{x\to+\infty} 1 + x^2 = +\infty$$

we obtain, by the Squeeze Theorem (rather, the “Push Theorem”) that $\lim_{x\to+\infty} f(x) = +\infty$. Hence $b = +\infty$.

c) Since sin is bounded and $\lim_{x\to+\infty} f(x) = +\infty$ we conclude, by (a corollary of) the Squeeze Theorem, that $\lim\limits_{x\to+\infty} \sin(x) + f(x) = +\infty$. Hence $c = +\infty$.

d) For $x \in \mathbb{R}_+^*$ (which is a punctured neighborhood of $+\infty$) we have:

$$\frac{1}{x^3} + \frac{1}{x} \leq \frac{f(x)}{x^3} \leq \frac{1}{x^3} + \frac{2}{x}.$$

Since

$$\lim_{x\to+\infty} \frac{1}{x^3} + \frac{1}{x} = \lim_{x\to+\infty} \frac{1}{x^3} + \frac{2}{x} = 0$$

we conclude, by the Squeeze Theorem, that $\lim\limits_{x\to+\infty} f(x) = 0$. Hence $d = 0$.

Exercise 4.

1. Let $x, y \in \mathbb{R}$ with $x < y \leq -2$. Since $y \leq -2$, by multiplying this inequality by x (with $x < 0$), we obtain $xy \geq -2x$. Now adding $x + y$ yields $x + y + xy \geq -x + y$. Since $x < y$, we have $-x + y > 0$ and hence $x + y + xy > 0$.

2. Let $x \in \mathbb{R} \setminus -1$. Now, from the following equivalence:

$$-x - 2 = -1 \iff -x = 1 \iff x = -1$$

we conclude (since $x \neq -1$) that $-x - 2 \neq -1$, hence $-x - 2$ is in the domain of f, hence $f(-x-2)$ is well-defined.

Now,

$$f(x) + 4 = \frac{2x^2 + 4x + 4}{1 + x}$$

and

$$f(-x-2) + 4 = -\frac{2x^2 + 4x + 4}{1 + x}.$$

Hence $f(-x-2) + 4 = -\big(f(x) + 4\big)$.

3. Let $x, y \in (-\infty, 2]$ such that $x < y$. Then

$$f(x) - f(y) = \frac{2(x-y)(x+y+xy)}{(1+x)(1+y)}.$$

Since $x - y < 0$, $x + y + xy > 0$ (by Question 1) and $1 + x < 0$ and $1 + y < 0$, we conclude that $f(x) - f(y) < 0$, hence $f(x) < f(y)$. Hence f is increasing on $(-\infty, 2]$.

4. For $x \in (-\infty, -1)$ (which is a neighborhood of $+\infty$),

$$\frac{2x^2}{1+x} = x\frac{2}{1 + 1/x} \underset{x\to-\infty}{\longrightarrow} -\infty \times \frac{2}{1+0} = -\infty.$$

Hence $\ell = -\infty$.

5. $J = (-\infty, -8]$. The -8 endpoint comes from the value of $f(2)$.

6. Since f is increasing on $(-\infty, -2]$, g is injective. Moreover, $g(I) = f(I) = J =$ codomain of g, so g is surjective. Hence g is a bijection.

7. $\inf(g) = \inf J = -\infty$, $\sup(g) = \sup J = -8$. $\min(g)$ doesn't exist since $\min(J)$ doesn't exist, and $\max(g) = \max(J) = -8$.

8. Let $x \in (-\infty, -2]$ and $y \in (-\infty, -8]$. Then:

$$g(x) = y \iff \frac{2x^2}{1+x} = y \iff 2x^2 - xy - y = 0.$$

We recognize a quadratic (in the variable x), the determinant of which is $\Delta = y^2 + 8y = y(y+8)$. Since $y \leq -8 < 0$, $\Delta \geq 0$, and the solutions of the quadratic are

$$\frac{y + \sqrt{y(y+8)}}{4} \quad \text{and} \quad \frac{y - \sqrt{y(y+8)}}{4}.$$

Among these solutions, only the second one, i.e.,

$$\frac{y-\sqrt{y(y+8)}}{4}.$$

belongs to $(-\infty,-2]$. Indeed,[1]

- $$\begin{aligned}\frac{y+\sqrt{y(y+8)}}{4}+2 &= \frac{y+8+\sqrt{-y}\sqrt{-(y+8)}}{4}\\ &= \frac{-\sqrt{-(y+8)}^2+\sqrt{-y}\sqrt{-(y+8)}}{4}\\ &= \frac{-\sqrt{-(y+8)}+\sqrt{-y}}{4}\sqrt{-(y+8)}\\ &\geq 0,\end{aligned}$$

  since $-y-8 \leq -y$ and hence $\sqrt{-(y+8)} \leq \sqrt{-y}$, so that

  $$\frac{y+\sqrt{y(y+8)}}{4} \geq -2.$$

  Moreover, the equality occurs if and only if $y=-8$ (in which case both solutions are equal to -2).

- $$\begin{aligned}\frac{y-\sqrt{y(y+8)}}{4}+2 &= \frac{y+8-\sqrt{-y}\sqrt{-(y+8)}}{4}\\ &= \frac{-\sqrt{-(y+8)}^2-\sqrt{-y}\sqrt{-(y+8)}}{4}\\ &= -\frac{\sqrt{-(y+8)}+\sqrt{-y}}{4}\sqrt{-(y+8)}\\ &\leq 0,\end{aligned}$$

  so that

  $$\frac{y-\sqrt{y(y+8)}}{4} \leq -2.$$

We hence conclude that:

$$\begin{aligned} g^{-1} \;:\; (-\infty,-8] &\longrightarrow (-\infty,-2]\\ y &\longmapsto \frac{y-\sqrt{y(y+8)}}{4}.\end{aligned}$$

9. Let $y \in (-\infty,-8]$ (which is a neighborhood of $-\infty$). Then,

$$\begin{aligned}\frac{y-\sqrt{y(y+8)}}{4} &= \frac{y-\sqrt{y^2(1+8/y)}}{4}\\ &= \frac{y-\sqrt{y^2}\sqrt{1+8/y}}{4}\\ &= \frac{y-|y|\sqrt{1+8/y}}{4}\\ &= \frac{y+y\sqrt{1+8/y}}{4}\\ &= y\frac{1+\sqrt{1+8/y}}{4}\\ &\underset{y\to-\infty}{\longrightarrow} = -\infty \times \frac{1}{4} = -\infty.\end{aligned}$$

The other limit is straightforward:

$$\lim_{y\to-8}\frac{y-\sqrt{y(y+8)}}{4} = \frac{-8}{4} = -2.$$

[1] notice that $y<0$ and $y+8\leq 0$, so that $\sqrt{y(y+8)} = \sqrt{-y}\sqrt{-y-8} = \sqrt{-y}\sqrt{-(y+8)}$.