INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 – S1 – Math Test #2 – 1h30

December 7, 2018

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1 (*Differential Equations*). The three questions of this exercise are independent from each other.

1. Let $y_0, v_0 \in \mathbb{R}$.

   a) Give the solution of the following initial value problem:

$$\text{(IVP)} \qquad \begin{cases} y'' - y' - 2y = 0 \\ y(0) = y_0 \\ y'(0) = v_0. \end{cases}$$

   b) Let y be the solution of Problem (IVP). Prove that[1] $\lim_{x\to+\infty} y(x) = 0$ if and only if $y_0 = -v_0$.

2. Give the general solution of the following differential equation:

$$(*) \qquad y'(x) - 3y(x) = 2\cos(3x).$$

3. Find a second order, linear, differential equation with constant coefficients that has the following general solution:

$$y(x) = \mathrm{e}^{-3x}\big(A\cos(2x) + B\sin(2x)\big) + 1.$$

Exercise 2 (*Hyperbolic Functions*). The goal of this exercise is to find the solution(s) (if any) of the following equation in $x \in \mathbb{R}$, by following a specific method.

$$(*) \qquad 4\cosh(x) + 5\sinh(x) = 6.$$

1. Recall the addition formula for sinh.
2. Explain why it's not possible to find $\alpha \in \mathbb{R}$ such that

$$\begin{cases} \sinh(\alpha) = 4 \\ \cosh(\alpha) = 5. \end{cases}$$

   Hint: use the Pythagorean Theorem.

   So we're quite sad that we can't write Equation (∗) as $\sinh(\alpha + x) = 6$ (because that would be "easy" to solve).

3. Find $\mu \in \mathbb{R}_+^*$ and $\alpha \in \mathbb{R}$ such that:

$$\forall x \in \mathbb{R}, \left(4\cosh(x) + 5\sinh(x) = 6 \iff \sinh(\alpha + x) = \frac{6}{\mu}\right).$$

4. Deduce all the solutions $x \in \mathbb{R}$ of Equation (∗). Simplify your answer as much as you can[2]

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[1] You're given the values of the following limits, in case you need them:

$$\lim_{x\to+\infty} \mathrm{e}^x = +\infty \quad \text{and} \quad \lim_{x\to-\infty} \mathrm{e}^x = 0.$$

[2] To help simplify your answer, we recall that:

$$\forall x \in \mathbb{R}, \ \operatorname{arcsinh}(x) = \ln\left(x + \sqrt{1+x^2}\right).$$

Exercise 3 (*Limits*). The questions of this exercise are independent from each other. The questions about computing limits are phrased as "compute the value of the following limit." If a limit you're asked to compute doesn't exist, prove that it doesn't exist.

1. Compute the value of the following limit.

$$\ell = \lim_{x\to 2} \frac{3}{x^3-3x-2} - \frac{1}{x^2-x-2}.$$

Hint: use the factored of the denominators to cross multiply the fractions (there are obvious roots!)

2. Let $f : \mathbb{R} \to [1,+\infty)$ be a function. Compute the value of the following limit.

$$\ell = \lim_{x\to+\infty} \frac{x\sqrt{2+x^2}}{(1+x^3)f(x)}.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\forall x \in \mathbb{R},\ 1+x^2 \leq f(x) \leq 1+2x^2.$$

a) Determine the value of the limit $a = \lim_{x\to 0} f(x)$.

b) Determine the value of the limit $b = \lim_{x\to+\infty} f(x)$.

c) Determine the value of the limit $c = \lim_{x\to+\infty} \sin(x) + f(x)$.

d) Determine the value of the limit $d = \lim_{x\to+\infty} \frac{f(x)}{x^3}$.

Exercise 4. We define the function f as

$$\begin{aligned} f : \mathbb{R}\setminus\{-1\} &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{2x^2}{1+x}. \end{aligned}$$

1. Preliminary question (that will be useful in Question 3 below): prove that for all $x, y \in \mathbb{R}$ such that $x < y \leq -2$ one has

$$x + y + xy > 0.$$

Hint: start with the inequality $y \leq -2$, multiply it by x (be careful with the sign of x) and then add $x + y$.

2. Let $x \in \mathbb{R}\setminus\{-1\}$. Compare $A = f(x) + 4$ and $B = f(-x-2) + 4$ (and, while you're at it, briefly justify why $f(-x-2)$ is defined). What can you conclude about the graph of f?

3. Determine (without using derivatives) the variations of f on $I = (-\infty, -2]$. You may use the result of Question 1.

4. Compute the value of the following limit (or if the limit doesn't exist, prove its non-existence):

$$\ell = \lim_{x\to-\infty} f(x).$$

5. Determine the interval $J = f(I)$ (no justifications required).

6. Briefly explain why the function g defined by

$$\begin{aligned} g : I &\longrightarrow J \\ x &\longmapsto f(x) \end{aligned}$$

is a bijection.

7. Determine $\inf(g)$ and $\sup(g)$. Do $\min(g)$ and $\max(g)$ exist?

8. Determine the function g^{-1} explicitly.

9. Compute the limits of g^{-1} at the endpoints of the interval J.