SCAN 1 — S1 — Solution of Math Test #3

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Exercise 1.

1. Let $n \in \mathbb{N}^*$.

$$\begin{aligned}
\frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) &= \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right) \\
&\leq \frac{1}{n+1} - \frac{1}{n+1} = 0, \\
\frac{1}{n+1} + \ln\left(\frac{n+2}{n+1}\right) &= \frac{1}{n+1} - \ln\left(1 + \frac{1}{n+1}\right) \\
&\geq \frac{1}{n+1} - \frac{1}{n+1} = 0,
\end{aligned}$$

where we used the given inequality with, in the first case, $x = -1/(n+1) \in (-1, +\infty)$ and, in the second case, $x = 1/(n+1) \in (-1, +\infty)$.

2. Let $n \in \mathbb{N}^*$. Then

$$u_{n+1} - u_n = -\ln(n+1) + \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n) - \sum_{k=1}^{n} \frac{1}{k} = \ln\left(\frac{n}{n+1}\right) + \frac{1}{n+1} \leq 0$$

and

$$\begin{aligned}
v_{n+1} - v_n &= u_{n+1} - u_n + \ln\left(\frac{n+1}{n+2}\right) - \ln\left(\frac{n}{n+1}\right) \\
&= \ln\left(\frac{n}{n+1}\right) + \frac{1}{n+1} + \ln\left(\frac{n+1}{n+2}\right) - \ln\left(\frac{n}{n+1}\right) \\
&= \frac{1}{n+1} + \ln\left(\frac{n+1}{n+2}\right) \\
&= \frac{1}{n+1} - \ln\left(\frac{n+2}{n+1}\right) \\
&\geq 0.
\end{aligned}$$

Hence the sequence $(u_n)_{n\in\mathbb{N}^*}$ is non-increasing and the sequence $(v_n)_{n\in\mathbb{N}^*}$ is non-decreasing.

3. From the previous question, we already know that the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ have opposite variations; so we only need to show that the limit of their difference is nil: let $n \in \mathbb{N}^*$. Then:

$$v_n - u_n = \ln\left(\frac{n}{n+1}\right) \underset{n\to+\infty}{\longrightarrow} \ln(1) = 0$$

(since ln is continuous at 1). Hence the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent sequences.

4. Since the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent, they converge to the same limit. In particular, we conclude that the limit γ exists, as the limit of the sequence $(u_n)_{n\in\mathbb{N}^*}$.

Moreover, from the variations of $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ we have:

$$v_1 \leq \gamma \leq u_1,$$

and the result follows from the values $u_1 = 1$ and $v_1 = 1 + \ln(1/2) = 1 - \ln(2)$.

Exercise 2.

1. arccos : $[-1, 1] \to [0, \pi]$, is decreasing, and its graph is given on Figure 2.

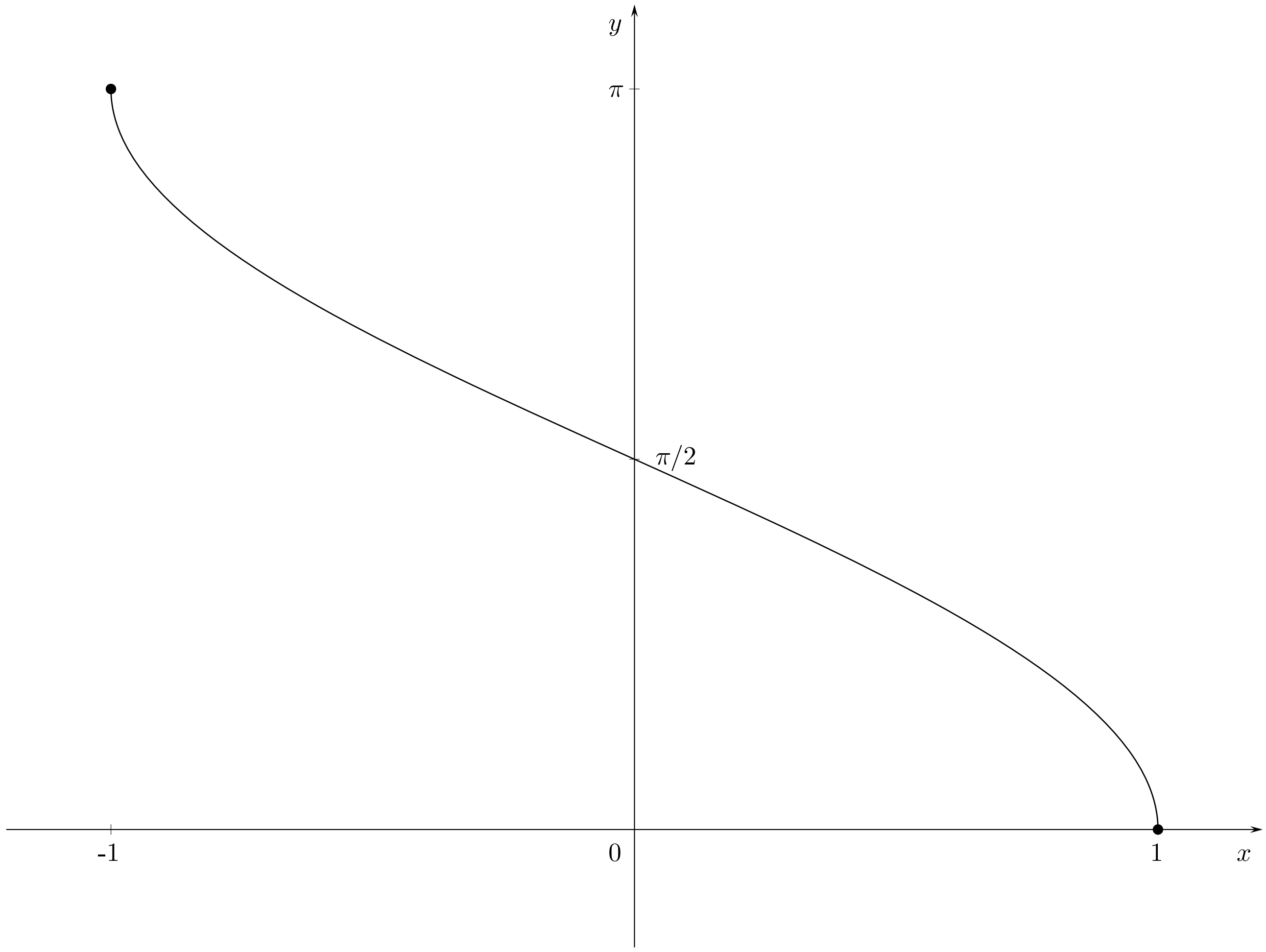

Figure 2 – Graph of the arccos function

2. Let $x \in \mathbb{R}$. Then:

$$\begin{aligned}
\arccos\big(e^x - 1\big) \text{ is defined } &\iff -1 \le e^x - 1 \le 1 \\
&\iff 0 \le e^x \le 2 \\
&\iff e^x \le 2 && \textit{since } e^x > 0 \\
&\iff x \le \ln(2) && \textit{since } \ln \textit{ is increasing..}
\end{aligned}$$

   Hence $D = (-\infty, \ln 2]$.

3. We know that exp is increasing, hence $x \mapsto e^x - 1$ is increasing; now composed with arccos (which is decreasing), we conclude that f is decreasing.

4. We know that $\lim\limits_{x\to-\infty} e^x - 1 = -1$, and that arccos is continuous at -1, hence

$$\ell = \lim_{x\to-\infty} f(x) = \arccos(-1) = \pi.$$

5. f is the composition of continuous functions, hence f is continuous. By (a corollary of) the Intermediate Value Theorem, and since f is decreasing, we have:

$$f(D) = \big[f(\ln 2), \ell\big) = \big[\arccos(1), \pi\big) = [0, \pi).$$

Exercise 3. First observe that for $n \in \mathbb{N}^*$, $p_n > 0$, as p_n is the product of positive terms. We can hence determine the variations of the sequence $(p_n)_{n\in\mathbb{N}^*}$ by using the ratio of two consecutive terms: for $n \in \mathbb{N}^*$,

$$\frac{p_{n+1}}{p_n} = 1 + \frac{1}{(n+1)^\alpha} > 1$$

hence the sequence $(p_n)_{n\in\mathbb{N}^*}$ is increasing. By the Monotone Limit Theorem, the limit $\lim\limits_{n\to+\infty} p_n$ exists in $\overline{\mathbb{R}}$.

Exercise 4.

1. Let I be a closed and bounded set (or interval) and let $f : I \to \mathbb{R}$ be a continuous function. Then f is bounded and attains its bounds.

2. See what we did in class...

Exercise 5.

1. We know that $\arctan : \mathbb{R} \to \mathbb{R}$ so that, for $x \in \mathbb{R}_+$, $\arctan(x)$ is well-defined; we also know that arctan is increasing and $\arctan(0) = 0$ hence, for $x \in \mathbb{R}_+$, $\arctan(x) \geq 0$; and hence $x \arctan(x) \geq 0$, and we conclude that f is well-defined.

2. Let $x \in \mathbb{R}_+^*$. Then:
$$C = \frac{f(x)}{x} + xf\left(\frac{1}{x}\right) = \frac{x\arctan(x)}{x} + x\frac{1}{x}\arctan\left(\frac{1}{x}\right) = \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$
Hence C is independent of x and its value is $\pi/2$.

3. a) Let $x, y \in \mathbb{R}_+$ such that $x < y$. We know that arctan is increasing, hence $0 \leq \arctan(x) < \arctan(y)$; and since $0 \leq x < y$, we have $0 \leq x\arctan(x) < y\arctan(y)$.
   b) Since arctan is continuous, we conclude that f is also continuous. Since f is increasing we conclude, by (a corollary of) the Intermediate Value Theorem, that
   $$f(\mathbb{R}_+) = \left[f(0), \lim_{+\infty} f\right).$$
   Clearly, $f(0) = 0$. Moreover, we know that $\lim\limits_{+\infty} \arctan = \pi/2$, and we hence conclude, by the elementary operations on limits that $\lim_{+\infty} f = +\infty$. Hence $f(\mathbb{R}_+) = [0, +\infty) = \mathbb{R}_+$, hence f is onto.
   c) f is increasing (hence injective) and onto, hence f is a bijection. Since f is a *continuous increasing bijection on an interval,* we conclude that f^{-1} is continuous.

Exercise 6.

1. a) For $x \in \mathbb{R}^*$:
   $$\frac{g(x)}{x} = \frac{\mathrm{e}^x - 1}{x} - \alpha \underset{x\to 0}{\longrightarrow} 1 - \alpha,$$
   hence $\ell_1 = 1 - \alpha < 0$, since $\alpha > 1$.
   b) By definition of the limit ℓ_1, for $\varepsilon = |\ell_1| = -\ell_1 > 0$, there exists $\delta > 0$ such that
   $$\forall x \in (-\delta, 0) \cup (0, \delta), \quad \left|\frac{g(x)}{x} - \ell_1\right| < -\ell_1.$$
   In particular, for $x \in (0, \delta)$,
   $$\frac{g(x)}{x} - \ell_1 < -\ell_1,$$
   i.e.,
   $$\frac{g(x)}{x} < 0.$$

2. For $x \in \mathbb{R}_+^*$,
$$\frac{g(x)}{x} = \frac{\mathrm{e}^x}{x} - \frac{1}{x} - \alpha \underset{x\to+\infty}{\longrightarrow} +\infty - 0 - \alpha = +\infty,$$
where we use the well-known limit $\lim\limits_{x\to+\infty} \mathrm{e}^x/x = +\infty$.

3. We use the Intermediate Value Theorem:
   - The function g is continuous on $\mathbb{R}$, and hence on $[\delta/2, M+1]$,
   - By Question 1.b), $g(\delta/2) < 0$,
   - By Question 2., $g(M+1) > 0$,

   hence, by the Intermediate Value Theorem, there exists $x_0 \in (\delta/2, M+1)$ such that $g(x_0) = 0$.