INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Math Test #3 — 1h30 *January 11, 2019*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1 (*Euler–Mascheroni Constant*). The goal of this exercise is to show that the following limit:

$$\gamma = \lim_{n\to+\infty}\left(-\ln(n) + \sum_{k=1}^{n}\frac{1}{k}\right)$$

exists in $\mathbb{R}$. The value of this limit is called the *Euler–Mascheroni* constant.
You're given the following inequality (and you may use it without any justifications):

$$\forall x \in (-1,+\infty),\ \ln(1+x) \le x.$$

1. Preliminary question: show that

$$\forall n \in \mathbb{N}^*,\ \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) \le 0$$

and that

$$\forall n \in \mathbb{N}^*,\ \frac{1}{n+1} - \ln\left(\frac{n+2}{n+1}\right) \ge 0.$$

We now define the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ by:

$$\forall n \in \mathbb{N}^*,\ u_n = -\ln(n) + \sum_{k=1}^{n}\frac{1}{k} \quad\text{and}\quad v_n = u_n + \ln\left(\frac{n}{n+1}\right).$$

2. Use the result of the preliminary question to determine the variations of the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$.
3. Show that the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent sequences.
4. Deduce that the limit γ exists, and explain why

$$1-\ln(2) \le \gamma \le 1.$$

Exercise 2. ✓

1. Recall (without any justifications) the domain, the range, and the graph of the function arccos.
2. Give the maximal subset D of $\mathbb{R}$ such that for all $x \in D$ the following expression is well-defined:

$$\arccos(e^x - 1).$$

Just to be sure: the set D you obtain should be a neighborhood of $-\infty$.

We hence define the function f as:

$$f : D \longrightarrow \mathbb{R},\quad x \longmapsto \arccos(e^x - 1).$$

3. Determine, as concisely as possible, the variations of f.
4. Compute the value of the limit:

$$\ell = \lim_{x\to-\infty}\arccos(e^x - 1).$$

Give all the necessary justifications.

5. Determine the range of f. Give all the necessary justifications.

Exercise 3. Let $\alpha \in \mathbb{R}_+^*$ and define the sequence $(p_n)_{n\in\mathbb{N}^*}$ as:

$$\forall n \in \mathbb{N}^*,\ p_n = \prod_{k=1}^{n}\left(1 + \frac{1}{k^\alpha}\right).$$

Explain why $\lim_{n\to+\infty} p_n$ exists in $\overline{\mathbb{R}}$.
Note that it's not recommended to try to compute the value of this limit explicitly, so you have to find another strategy to show that the limit exists.

Exercise 4.

1. Recall the Extreme Value Theorem.
2. Give *one* counter example of the Extreme Value Theorem, that is, show that if you drop one of the hypotheses of the theorem, it's possible to find a counter-example.[3]

Exercise 5. *In this exercise, questions 2 and 3 are independent of each other.*
We define the function:

$$\begin{array}{rccl} f: & \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ & x & \longmapsto & x\arctan(x). \end{array}$$

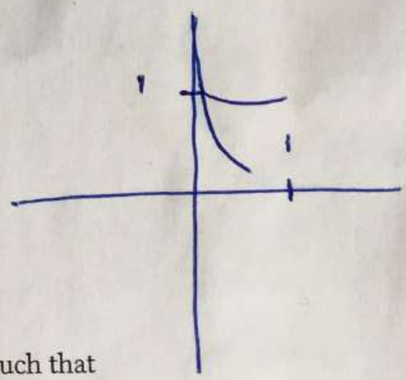
1. Briefly justify that f is well-defined.
2. Show that the expression
$$C = \frac{f(x)}{x} + xf\left(\frac{1}{x}\right)$$
is independent of $x \in \mathbb{R}_+^*$ (i.e., C is constant with respect to x), and determine the value of C.
3. a) Show that f is increasing.
   b) Show that f is onto.
   c) Explain why f is a bijection and why f^{-1} is continuous.

Exercise 6. Let $\alpha \in (1, +\infty)$.
The goal of this exercise is to show that the equation

$$(E_\alpha) \qquad e^x = \alpha x + 1$$

possesses at least one solution in $\mathbb{R}_+^*$.
We define the function g as:

$$\begin{array}{rccl} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & e^x - \alpha x - 1. \end{array}$$



1. a) Determine the value of the following limit
$$\ell_1 = \lim_{x\to 0}\frac{g(x)}{x},$$
as well as the sign of ℓ_1 (just to be sure, $\ell_1 \in \mathbb{R}^*$)
   b) Use the ε-δ definition of the limit ℓ_1 with $\varepsilon = |\ell_1|$ to show that there exists $\delta > 0$ such that
$$\forall x \in (0, \delta),\ g(x) < 0.$$

2. Determine the value of the following limit:
$$\ell_2 = \lim_{x\to+\infty}\frac{g(x)}{x},$$
As in question 1b, we could show that there exists $M > 0$ such that
$$\forall x \in (M, +\infty),\ g(x) > 0.$$

3. Deduce that the equation (E_α) possesses at least one solution in $\mathbb{R}_+^*$.

[3] In class we did all the possible cases, you're only asked to do one of them; that is, you're free to choose the hypothesis you drop; specify this explicitly. Give an explicit counter example, and the corresponding graph.