SCAN 1 — S1 — Solution of Math Test #4

March 29, 2019

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Exercise 1.

1. The function sin is of class C^6 on $[0,1/4]$ and 7 times differentiable on $(0,1/4)$ hence, by the Taylor–Lagrange formula, there exists $c\in(0,1/4)$ such that

$$\begin{aligned}\sin(1/4)&=\sum_{k=0}^{6}\frac{\sin^{(k)}(0)}{k!}\left(\frac14\right)^k+\frac{\sin^{(7)}(c)}{7!}\left(\frac14\right)^7\\&=\frac14-\frac16\left(\frac14\right)^3+\frac{1}{120}\left(\frac14\right)^5-\frac{\cos(c)}{5040}\left(\frac14\right)^7.\end{aligned}$$

   Now since $0<c<1/4<\pi/2$ and since cos is decreasing on $[0,\pi/2]$, $1>\cos(c)>0$, hence $-1<-\cos(c)<0$ and hence

$$-\frac{1}{5040}\left(\frac14\right)^7<-\frac{\cos(c)}{5040}\left(\frac14\right)^7<0$$

   hence

$$\frac14-\frac16\left(\frac14\right)^3+\frac{1}{120}\left(\frac14\right)^5-\frac{1}{5040}\left(\frac14\right)^7<\sin\left(\frac14\right)<\frac14-\frac16\left(\frac14\right)^3+\frac{1}{120}\left(\frac14\right)^5.$$

2. From the values given by the calculator, we conclude:

$$\underline{0.2474039}5<0.247403959244016617\,\overline{063492}<\sin(1/4)<0.2474039713541\,\overline{6}<\underline{0.2474039}8.$$

   Hence,

$$\sin\left(\frac14\right)=0.2474039\ldots$$

Exercise 2.

1. By long divisions we obtain:

$$\frac{1}{\sin(x)}\underset{x\to0}{=}\frac1x+\frac x6+o\big(x^2\big),\qquad\qquad \frac{1}{\ln(1+x)}\underset{x\to0}{=}\frac1x+\frac12-\frac{x}{12}+\frac{x^2}{24}+o\big(x^2\big),$$

   hence

$$f(x)\underset{x\to0}{=}=-\frac12+\frac x4-\frac{x^2}{24}+o\big(x^2\big).$$

   We conclude that $\lim\limits_{x\to0}f(x)=-1/2\in\mathbb{R}$, hence f possesses an extension by continuity at 0:

$$\begin{aligned}\tilde f\ :\ (-1,\pi)&\longrightarrow\mathbb{R}\\ x&\longmapsto\begin{cases}f(x)&\text{if }x\neq0\\-1/2&\text{if }x=0.\end{cases}\end{aligned}$$

   Moreover, $\tilde f$ possesses a first order Taylor–Young expansion at 0, hence $\tilde f$ is differentiable at 0, and $\tilde f'(0)=1/4$. An equation of Δ is:

$$\Delta\colon y=-\frac12+\frac x4.$$

   Since the quadratic term of the Taylor–Young expansion of $\tilde f$ is negative we conclude that the graph of $\tilde f$ lies below Δ in a neighborhood of 0.

2. See Figure 3.

Exercise 3.

1. For $n\in\mathbb{N}^*$,

$$\left(1+\frac1n\right)^n-\mathrm{e}=\exp\left(n\ln\left(1+\frac1n\right)\right)-\mathrm{e}$$

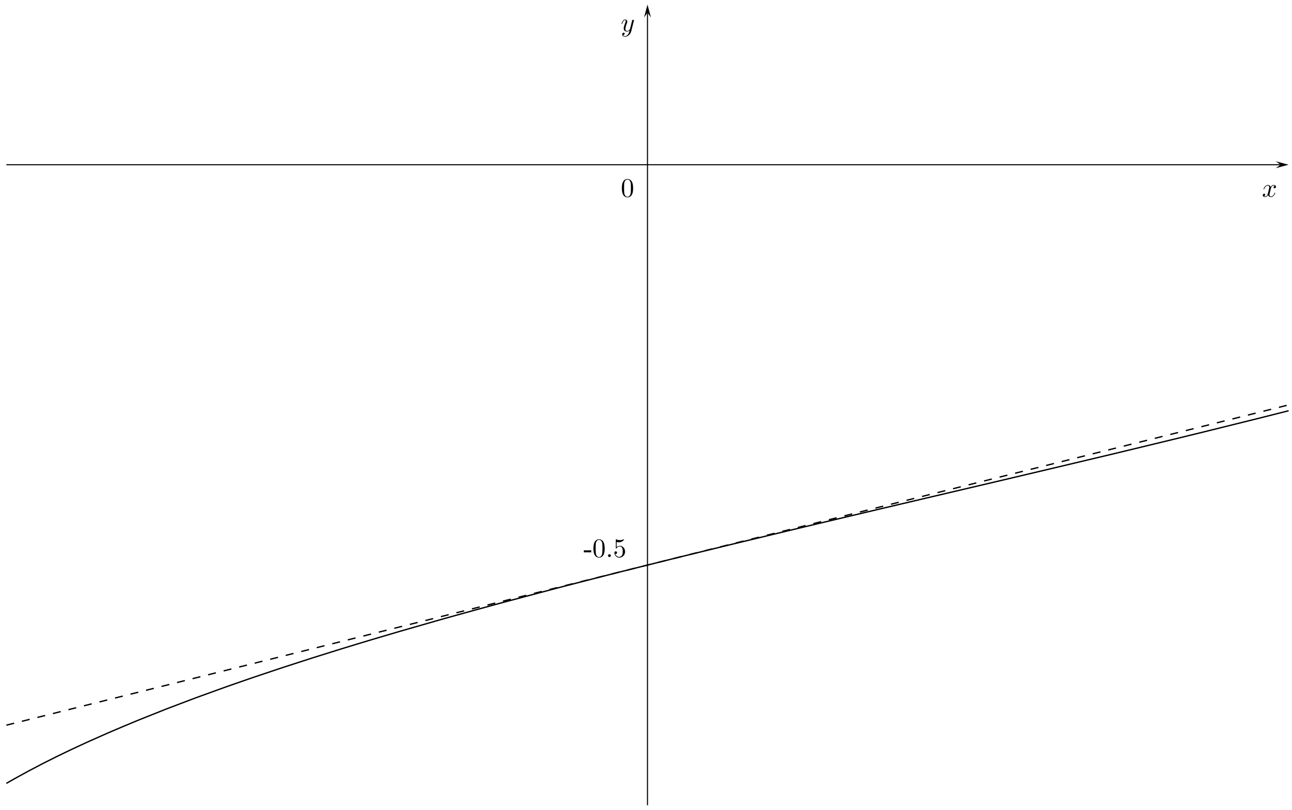

Figure 3 – Graph of function $\tilde{f}$ of Exercise 2 (plain) and its tangent line Δ at $(0, -1/2)$ (dashed).

$$= \mathrm{e}\left(\exp\left(n\ln\left(1+\frac{1}{n}\right)-1\right)-1\right)$$
$$\underset{n\to+\infty}{\sim} \mathrm{e}\left(n\ln\left(1+\frac{1}{n}\right)-1\right) \qquad \textit{since } n\ln\left(1+\frac{1}{n}\right)-1 \underset{n\to+\infty}{\longrightarrow} 0$$

Now,

$$\ln\left(1+\frac{1}{n}\right) \underset{n\to+\infty}{=} \frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right),$$

hence

$$n\ln\left(1+\frac{1}{n}\right)-1 \underset{n\to+\infty}{=} -\frac{1}{2n}+o\left(\frac{1}{n}\right) \underset{n\to+\infty}{\sim} -\frac{1}{2n},$$

and we conclude:

$$\left(1+\frac{1}{n}\right)^n-\mathrm{e} \underset{n\to+\infty}{\sim} -\frac{\mathrm{e}}{2n}.$$

2. For $x \in \mathbb{R}_+^*$,

$$\begin{aligned}\left(1+x^x\right)^{1/x}-x &= x\left(\left(1+x^{-x}\right)^{1/x}-1\right)\\ &= x\left(\exp\left(\frac{1}{x}\ln\left(1+\frac{1}{x^x}\right)\right)-1\right)\\ &\underset{x\to+\infty}{\sim} x\left(\frac{1}{x}\ln\left(1+\frac{1}{x^x}\right)\right) \qquad \textit{since } \frac{1}{x}\ln\left(1+\frac{1}{x^x}\right) \underset{x\to+\infty}{\longrightarrow} 0\\ &= \ln\left(1+\frac{1}{x^x}\right) \underset{x\to+\infty}{\sim} \frac{1}{x^x}.\end{aligned}$$

3. Using the composition of the usual Taylor–Young expansions, we obtain:

$$\ln\big(\cos(x)\big) \underset{x\to0}{=} -\frac{x^2}{2}-\frac{x^4}{12}+o\big(x^4\big),$$

hence

$$\ln\big(\cos(x)\big) + \frac{x^2}{2} \underset{x\to 0}{=} -\frac{x^4}{12} + o\big(x^4\big) \underset{x\to 0}{\sim} -\frac{x^4}{12}.$$

Exercise 4. By dividing the Taylor–Young expansions of sin and cos, we obtain:

$$\tan(x) \underset{x\to 0}{=} x + \frac{x^3}{3} + \frac{2}{15}x^5 + o\big(x^5\big).$$

Hence,

$$\tan(x) - x - \frac{x^3}{3} \underset{x\to 0}{=} \frac{2}{15}x^5 + o\big(x^5\big) \underset{x\to 0}{\sim} \frac{2}{15}x^5.$$

Hence,

$$\frac{\tan(x) - x - x^3/3}{\big(\cos(x) - 1\big)\sin^3(x)} \underset{x\to 0}{\sim} \frac{2x^5/15}{-x^2/2 \times x^3} = -\frac{4}{15} \underset{x\to 0}{\longrightarrow} -\frac{4}{15}$$

Exercise 5.

1. The function f is differentiable on $\mathbb{R}$ and for $x \in \mathbb{R}$ one has:

   $$f'(x) = \mathrm{e}^x\big(\cos(x) - \sin(x)\big).$$

   We look for the zeroes of f': for $x \in \mathbb{R}$,

   $$f'(x) = 0 \iff \cos(x) = \sin(x) \iff \exists k \in \mathbb{Z},\ x = \frac{\pi}{4} + k\pi.$$

   It is now easy to deduce that

   - $\forall x \in (-3\pi/4, \pi/4),\ f'(x) > 0$,
   - $\forall x \in (-7\pi/4, -3\pi/4) \cup (\pi/4, 5\pi/4),\ f'(x) < 0$,

   hence $I = (-3\pi/4, \pi/4)$.

   We can also determine J, using (a corollary of) the Intermediate Value Theorem:

   $$J = f(I) = \big[f(-3\pi/4), f(\pi/4)\big] = \left[-\frac{\sqrt{2}}{2}\mathrm{e}^{-3\pi/4}, \frac{\sqrt{2}}{2}\mathrm{e}^{\pi/4}\right].$$

2. By the Inverse Function Theorem, since g is an increasing bijection, differentiable on I and such that $\forall x \in I,\ g'(x) \neq 0$, we conclude that g^{-1} is differentiable on I. Moreover,

   $$\forall y \in J,\ \big(g^{-1}\big)'(y) = \frac{1}{g'\big(g^{-1}\big)(y)} = \frac{1}{\mathrm{e}^{g^{-1}(y)}\Big(\cos\big(g^{-1}(y)\big) - \sin\big(g^{-1}(y)\big)\Big)}.$$

3. By multiplying the usual Taylor–Young expansions of exp and cos at 0 we obtain:

   $$g(x) \underset{x\to 0}{=} 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + o\big(x^4\big).$$

4. Since we're given that g^{-1} is of class C^∞ on J and $1 \in J$, we conclude that g^{-1} is 4 times differentiable at 1. Hence, by the Taylor–Young Theorem, g^{-1} possesses a 4-th order Taylor–Young expansion at 1 of the form

   $$g^{-1}(y) \underset{y\to 1}{=} r + s(y-1) + t(y-1)^2 + u(y-1)^3 + v(y-1)^4 + o\big((y-1)^4\big).$$

5. We know that $r = g^{-1}(1)$ and $s = \big(g^{-1}\big)'(1)$. Now, since $g(0) = 1$ we conclude that $r = g^{-1}(1) = 0$. Also,

   $$s = \big(g^{-1}\big)'(1) = \frac{1}{\mathrm{e}^0\big(\cos(0) - \sin(0)\big)} = 1.$$

   At this point we have:

   $$g^{-1}(y) \underset{y\to 1}{=} (y-1) + t(y-1)^2 + u(y-1)^3 + v(y-1)^4 + o\big((y-1)^4\big).$$

6.

$$
\begin{aligned}
y = 1+(y-1) = g\big(g^{-1}(y)\big) &\underset{y\to 1}{=} 1 + g^{-1}(y) - \frac{\big(g^{-1}(y)\big)^3}{3} - \frac{\big(g^{-1}(y)\big)^4}{6} + o\Big(\big(g^{-1}(y)\big)^4\Big)\\
&\underset{y\to 1}{=} 1 + (y-1) + t(y-1)^2 + u(y-1)^3 + v(y-1)^4 + o\big((y-1)^4\big)\\
&\quad - \frac{1}{3}\Big((y-1) + t(y-1)^2 + u(y-1)^3 + v(y-1)^4 + o\big((y-1)^4\big)\Big)^3\\
&\quad - \frac{1}{6}\Big((y-1) + t(y-1)^2 + u(y-1)^3 + v(y-1)^4 + o\big((y-1)^4\big)\Big)^4\\
&\quad + o\Big(\big(g^{-1}(y)\big)^4\Big)\\
&\underset{y\to 1}{=} 1 + (y-1) + t(y-1)^2 + u(y-1)^3 + v(y-1)^4\\
&\quad - \frac{1}{3}(y-1)^3 - \frac{1}{6}(y-1)^4 + o\big((y-1)^4\big) \qquad\qquad \textit{since } g^{-1}(y) \underset{y\to 1}{\sim} (y-1)\\
&\underset{x\to 1}{=} 1 + (y-1) + t(y-1)^2 + \left(u - \frac{1}{3}\right)(y-1)^3 + \left(v - \frac{1}{6}\right)(y-1)^4 + o\big((y-1)^4\big).
\end{aligned}
$$

Using the uniqueness of a Taylor–Young expansion, and identifying this term with $y = 1 + (y-1)$, we obtain:

$$
t = 0, \qquad u = \frac{1}{3}, \qquad \text{and} \qquad v = \frac{1}{6}.
$$

Hence,

$$
g^{-1}(y) \underset{y\to 1}{=} (y-1) + \frac{1}{3}(y-1)^3 + \frac{1}{6}(y-1)^4 + o\big((y-1)^4\big).
$$

Exercise 6.

1. We know that the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing and bounded from above (by 1) hence, by the Monotone Limit Theorem, the sequence $(u_n)_{n\in\mathbb{N}^*}$ converges, say $\ell = \lim\limits_{n\to+\infty} u_n \in [1/\mathrm{e}, 1]$.

   Now, from the definition of the sequence $(u_n)_{n\in\mathbb{N}^*}$,

   $$\forall n \in \mathbb{N}^*,\ f(u_n) = -\frac{1}{n},$$

   and since f is continuous at $\ell \in [1/\mathrm{e}, 1] \subset \mathbb{R}_+^*$ we must have (by taking the limit as $n \to +\infty$):

   $$f(\ell) = 0,$$

   i.e., $\ell \ln(\ell) = 0$, hence (since $\ell \neq 0$), $\ln(\ell) = 0$ from which we deduce that $\ell = 1$.

2. a) Since $u_n \underset{n\to+\infty}{\longrightarrow} 1$, we have:

   $$u_n \ln(u_n) \underset{n\to+\infty}{\sim} u_n - 1 = v_n,$$

   where we used the well-known equivalent $\ln(X) \underset{X\to 1}{\sim} X - 1$.

   b) We hence have:

   $$-\frac{1}{n} = f(u_n) \underset{n\to+\infty}{\sim} v_n = u_n - 1,$$

   i.e.,

   $$\lim_{n\to+\infty} -n(u_n - 1) = 1,$$

   hence

   $$\lim_{n\to+\infty} -n(u_n - 1) - 1 = 0,$$

   hence

   $$\lim_{n\to+\infty} n\left(u_n - 1 + \frac{1}{n}\right) = 0,$$

   i.e.,

   $$u_n - 1 + \frac{1}{n} \underset{n\to+\infty}{=} o\left(\frac{1}{n}\right),$$

   as required.

3. Let $n \in \mathbb{N}^*$. From $f(u_n) = -1/n$ we obtain $u_n \ln(u_n) = -1/n$ hence (since $u_n \neq 0$), $\ln(u_n) = -1/nu_n$ hence

$$u_n = \exp\left(-\frac{1}{nu_n}\right).$$

Now

$$nu_n \underset{n\to+\infty}{=} n\left(1 - \frac{1}{n} + o\left(\frac{1}{n}\right)\right) \underset{n\to+\infty}{=} n - 1 + o(1)$$

hence,

$$\begin{aligned}
-\frac{1}{nu_n} &\underset{n\to+\infty}{=} -\frac{1}{n-1+o(1)} \\
&\underset{n\to+\infty}{=} -\frac{1}{n}\left(\frac{1}{1-1/n+o(1/n)}\right) \\
&\underset{n\to+\infty}{=} -\frac{1}{n}\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right) \\
&\underset{n\to+\infty}{=} -\frac{1}{n} - \frac{1}{n^2} + o\left(\frac{1}{n^2}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
u_n = \exp\left(-\frac{1}{nu_n}\right) &\underset{n\to+\infty}{=} 1 + \left(-\frac{1}{nu_n}\right) + \frac{1}{2}\left(-\frac{1}{nu_n}\right)^2 + o\left(\left(\frac{1}{nu_n}\right)^2\right) \\
&\underset{n\to+\infty}{=} 1 + \left(-\frac{1}{n} - \frac{1}{n^2}\right) + \frac{1}{2}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) \\
&\underset{n\to+\infty}{=} 1 - \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right).
\end{aligned}$$

Hence $a = -1/2$.

Exercise 7. The value of the Riemann sum is:

$$R = 0.1^2 \times 0.5 + 0.7^2 \times 0.3 + 0.9^2 \times 0.2 = 0.314.$$

The exact value of I is (using the Fundamental Theorem of Calculus):

$$I = \int_0^1 t^2 \,\mathrm{d}t = \left[\frac{t^3}{3}\right]_{t=0}^{t=1} = \frac{1}{3}.$$