INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #2 — 1h30

November 29, 2019

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1.

1. Give the solution of the following initial value problem:

$$(*)\qquad \begin{cases} f'' + 2f' + 10f = 2 \\ f(0) = 1 \\ f'(0) = -1. \end{cases}$$

2. Give the general solution of the following differential equation:

$$\text{(E)}\qquad \forall t \in \mathbb{R},\ f'(t) - 2f(t) = 4\cos(2t).$$

Exercise 2.

1. Prove that

$$\forall x \in \mathbb{R},\ 1 - \tanh^2(x) = \frac{1}{\cosh^2(x)}.$$

2. Find the maximal subset D of $\mathbb{R}$ such that for all $x \in D$, the expression

$$\tanh\big(\operatorname{arccosh}(x)\big)$$

is well defined.

3. Let $x \in D$. Simplify the expression $\tanh\big(\operatorname{arccosh}(x)\big)$ as much as possible.

Exercise 3. Define the function f by:

$$f : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{1 + \tanh(x)}. \end{array}$$

1. Explain why f is well-defined.
2. Determine the variations of f.
3. Sketch the graph of f.
4. Determine (without any justifications), if they exist, the value of

$$\sup_{\mathbb{R}} f, \qquad \max_{\mathbb{R}} f, \qquad \inf_{\mathbb{R}} f, \qquad \min_{\mathbb{R}} f.$$

In case of non-existence, write "DNE."

Exercise 4. We define the function

$$f : \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{\sqrt{x}+2}{3}. \end{array}$$

Let $u_0 \in (0,1)$ and define the sequence $(u_n)_{n\in\mathbb{N}}$ by:

$$\forall n \in \mathbb{N},\ u_{n+1} = f(u_n).$$

You're given that the sequence $(u_n)_{n\in\mathbb{N}}$ is well defined.

1. Preliminary question: give the factored form of the polynomial function P defined by:

$$\forall x \in \mathbb{R},\ P(x) = -x^2 + \frac{1}{3}x + \frac{2}{3}.$$

   Deduce that $\forall x \in (0,1),\ P(x) > 0$.

2. What are the variations of f?
3. Show that $f\big((0,1)\big) \subset (0,1)$.
4. Deduce that $\forall n \in \mathbb{N},\ u_n \in (0,1)$.
5. Show that $(u_n)_{n\in\mathbb{N}}$ is increasing. You may use the result of Question 1.
6. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ converges and determine its limit ℓ.

Exercise 5. Compute the value of the following limits:

$$\ell_1 = \lim_{x\to 1} \frac{\ln\big(x^2\big)}{x^2-3x+2} \qquad\qquad \ell_2 = \lim_{x\to 0} \frac{\cosh(x)-1}{x^2}.$$

For ℓ_2, you may use, without any justifications, the following facts:

$$\lim_{x\to 0} \frac{\sinh(x)}{x} = 1 \qquad \textit{and} \qquad \lim_{x\to 0} \cosh(x) = 1.$$