INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

# SCAN 1 — Solution of Math Test #3

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**Exercise 1.** Define the function

$$h\ :\ \begin{array}{rcl}[0,1] & \longrightarrow & \mathbb{R}\\ x & \longmapsto & f(x)-g(x).\end{array}$$

We apply the Intermediate Value Theorem to $h$:

- $h$ is continuous since $f$ and $g$ are continuous,
- $h(0)=f(0)-g(0)=-g(0)\leq 0$ since $g(0)\in[0,1]$,
- $h(1)=f(1)-g(1)=1-g(1)\geq 0$ since $g(1)\in[0,1]$,

hence, by the Intermediate Value Theorem, there exists $x_0\in[0,1]$ such that $h(x_0)=0$, i.e., such that $f(x_0)=g(x_0)$.

**Exercise 2.**

1. - Clearly, $g_0(0)=1$, and if $N\in\mathbb{N}^*$,

   $$g_N(0)=1+\sum_{k=1}^{N}\frac{x^k}{k!}=1+0=1.$$

   Hence, $\forall N\in\mathbb{N},\ g_N(0)=1$.
   - Let $N\in\mathbb{N}^*$ and $x\in\mathbb{R}$. Then:

   $$\begin{aligned}g_N'(x)&=\frac{\mathrm{d}}{\mathrm{d}x}\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^N}{N!}\right)\\&=0+1+x+\frac{x^2}{2}+\cdots+N\frac{x^{N-1}}{N!}\\&=1+x+\frac{x^2}{2}+\cdots+\frac{x^{N-1}}{(N-1)!}\\&=g_{N-1}(x).\end{aligned}$$

   Hence $g_N'=g_{N-1}$.
   - Let $N\in\mathbb{N}^*$. By the product rule we obtain, for $x\in\mathbb{R}$:

   $$\begin{aligned}f_N'(x)&=-\mathrm{e}^{-x}g_N(x)+\mathrm{e}^{-x}g_N'(x)=\mathrm{e}^{-x}\big(g_N'(x)-g_N(x)\big)\\&=\mathrm{e}^{-x}\big(g_{N-1}(x)-g_N(x)\big)\\&=-\mathrm{e}^{-x}\frac{x^N}{N!}.\end{aligned}$$

   - For $N=0$,

   $$f_0'(x)=-\mathrm{e}^{-x}=-\mathrm{e}^{-x}\frac{x^0}{0!}.$$

   Hence the statement is also true for $N=0$. So that:

   $$\forall N\in\mathbb{N},\ \forall x\in\mathbb{R},\ f_N'(x)=-\mathrm{e}^{-x}\frac{x^N}{N!}.$$

2. Let $N\in\mathbb{N}^*$ and $x\in\mathbb{R}$. Then:

   $$f_N''(x)=\mathrm{e}^{-x}\frac{x^N}{N!}-\mathrm{e}^{-x}N\frac{x^{N-1}}{N!}=\mathrm{e}^{-x}\left(\frac{x^N}{N!}-\frac{x^{N-1}}{(N-1)!}\right)=\mathrm{e}^{-x}\frac{x^{N-1}}{(N-1)!}\left(\frac{x}{N}-1\right).$$

   Now if $x\in(0,1)$, $x/N-1<0$, and all the other factors in $f_N''(x)$ are positive, hence $f_N''(x)<0$, and we conclude (since $[0,1]$ is an interval) that $f_N'$ is decreasing on $[0,1]$.

   The function $f_0'$ is clearly increasing.

3. Let $N \in \mathbb{N}^*$. We know that:
   - $f_N$ is continuous on $[0,1]$,
   - $f_N$ is differentiable on $(0,1)$,

   hence, by the Mean Value Theorem, there exists $c \in (0,1)$ such that
   $$f_N(1) - f_N(0) = f_N'(c).$$
   Since $f_N'$ is decreasing and since $0 < c < 1$, we conclude that:
   $$f_N'(1) < f_N(1) - f_N(0) < f_N'(0).$$
   Hence:
   $$-\frac{\mathrm{e}^{-1}}{N!} < \mathrm{e}^{-1} g_N(1) - 1 < 0,$$
   hence (multiplying by $\mathrm{e} > 0$):
   $$-\frac{1}{N!} < g_N(1) - \mathrm{e} < 0,$$
   hence the result.
4. Since $1/N! \underset{N\to+\infty}{\longrightarrow} 0$ we conclude, by the Squeeze Theorem, that $g_N(1) \underset{N\to+\infty}{\longrightarrow} \mathrm{e}$, as required.
5. From the numerical values given, we conclude that
   $$2.7182 < g_7(1) < 2.7183,$$
   We now use the inequality obtained in Question 2 in the case $N = 7$, namely,
   $$g_7(1) < \mathrm{e} < g_7(1) + \frac{1}{7!},$$
   and we deduce:
   $$2.7182 < g_7(1) < \mathrm{e} < g_7(1) + \frac{1}{7!} < 2.7183 + 0.0002 = 2.7185.$$
   Hence $\mathrm{e} = 2.718\ldots$

**Exercise 3.**

1. We know that arcsin and arccos are differentiable on $(-1,1)$ hence, by the product rule, $f$ is differentiable on $(-1,1)$ and:
   $$\forall x \in (-1,1),\ f'(x) = \frac{1}{\sqrt{1-x^2}}\big(\arccos(x) - \arcsin(x)\big).$$
2. Let $x \in (-1,1)$. Then the sign of $f'(x)$ is that of $\arccos(x) - \arcsin(x) = \pi/2 - 2\arcsin(x)$. Since arcsin is increasing and since $\arcsin(1/\sqrt{2}) = \pi/4$ we conclude that
   $$\forall x \in (-1, 1/\sqrt{2}),\ f'(x) > 0 \qquad \text{and} \qquad \forall x \in (1/\sqrt{2}, 1),\ f'(x) < 0.$$
   Hence $f$ is increasing on $[-1, 1/\sqrt{2}]$ and decreasing on $[1/\sqrt{2}, 1]$.

**Exercise 4.**

1. We rewrite $f$ using the exponential:
   $$\forall x \in \mathbb{R},\ f(x) = \exp\Big(x\ln\big(1+x^2\big)\Big),$$
   and we compute $f'$ using the chain rule and the product rule:
   $$\forall x \in \mathbb{R},\ f'(x) = \left(\ln\big(1+x^2\big) + \frac{2x^2}{1+x^2}\right)\big(1+x^2\big)^x.$$

2. For $x \in \mathbb{R}$,

$$g'(x) = \frac{\cosh(x)}{1+\sinh^2(x)} - \frac{2\mathrm{e}^x}{1+\mathrm{e}^{2x}} = \frac{\cosh(x)}{\cosh^2(x)} - \frac{2}{\mathrm{e}^{-x}+\mathrm{e}^x} = \frac{1}{\cosh(x)} - \frac{1}{\cosh(x)} = 0.$$

Since $\mathbb{R}$ is an interval, we conclude that $g$ is constant. We determine the value of $g$ by evaluating at 0: $g(0) = -\pi/2$, and we conclude that

$$\arctan\circ\sinh = -\frac{\pi}{2} + 2\arctan\circ\exp.$$

**Exercise 5.** Let $x \in \mathbb{R}^*$. Then:

$$\begin{aligned}\frac{(fg)(x)-(fg)(0)}{x} &= \frac{f(x)g(x)}{x} && \textit{since } f(0)=0\\ &= \frac{f(x)-f(0)}{x}g(x) && \textit{since } f(0)=0\\ &\underset{x\to 0}{\longrightarrow} 0,\end{aligned}$$

since $g$ is bounded and

$$\frac{f(x)-f(0)}{x} \underset{x\to 0}{\longrightarrow} f'(0) = 0.$$

**Exercise 6.** We know that $f$ is an increasing bijection and that $f$ is differentiable. Let $y \in \mathbb{R}$. By the inverse function rule, we know that $f^{-1}$ is differentiable at $y$ if and only if $f'(f^{-1}(y)) \neq 0$. Now,

$$\forall x \in \mathbb{R},\ f'(x) = x^2 - 2x + 1 = (x-1)^2,$$

hence $f^{-1}$ is differentiable at $y$ if and only if $f^{-1}(y) \neq 1$, if and only if $y \neq f(1) = \frac{1}{3}$. Hence, $D = \mathbb{R}\setminus\{1/3\}$. For $x \in f^{[-1]}(D)$,

$$\left(f^{-1}\right)'\big(f(x)\big) = \frac{1}{f'(x)} = \frac{1}{(x-1)^2}.$$

**Exercise 7.** Using the well-known equivalents:

$$\begin{aligned}\arcsin(x) &\underset{x\to 0}{\sim} \arctan(x) \underset{x\to 0}{\sim} x,\\ (1+X)^{1/3}-1 &\underset{X\to 0}{\sim} \frac{X}{3},\\ \cos(x)-1 &\underset{x\to 0}{\sim} -\frac{x^2}{2},\\ \cosh(X)-1 &\underset{X\to 0}{\sim} \frac{X^2}{2},\end{aligned}$$

we obtain, by product, quotient and substitution:

$$A(x) \underset{x\to 0}{\sim} \frac{x^2 x\dfrac{(-x^2)}{2}}{\dfrac{\sin(x)}{3}\left(\dfrac{(2x)^2}{2}\right)^2} = -\frac{3x}{8\sin(x)} \underset{x\to 0}{\longrightarrow} -\frac{3}{8}.$$

**Exercise 8.**

$$\frac{f+g}{g} = \frac{f}{g} + 1 \underset{a}{\longrightarrow} 0+1,$$

since $f/g \underset{a}{\longrightarrow} 0$ since $f \underset{a}{=} o(g)$.