INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #3 — 1h30

January 10, 2020

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be two continuous functions such that $f(0) = 0$ and $f(1) = 1$. Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

Exercise 2 (*Numerical approximation of* e)**.** For $N \in \mathbb{N}$ we define:

$$g_N : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\sum_{k=0}^{N} \frac{x^k}{k!} \end{array} \qquad \text{and} \qquad f_N : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \mathrm{e}^{-x} g_N(x). \end{array}$$

1. Let $N \in \mathbb{N}$. Give the value of $g_N(0)$.
2. Let $N \in \mathbb{N}^*$. Express g_N' in terms of g_{N-1} and show that

   $$\forall x \in \mathbb{R},\ f_N'(x) = -\mathrm{e}^{-x}\,\frac{x^N}{N!}.$$

   Is this last statement true in the case $N = 0$?
3. Let $N \in \mathbb{N}^*$. Give an expression of f_N'', and deduce the variations of f_N' on $[0,1]$. What are the variations of f_0' on $[0,1]$?
4. Let $N \in \mathbb{N}^*$. Apply the Mean Value Theorem to f_N on $[0,1]$, and deduce that

   $$\mathrm{e} - \frac{1}{N!} < g_N(1) < \mathrm{e}.$$

5. Deduce that

   $$\mathrm{e} = \lim_{N \to +\infty} \sum_{k=0}^{N} \frac{x^k}{k!}.$$

6. You're given that:

   $$g_7(1) = \frac{685}{252} = 2.71\overline{825396} \qquad \text{and} \qquad \frac{1}{7!} = \frac{1}{5040} < \frac{1}{5000} = 0.0002,$$

   (where the bar over the digits means that these digits are repeated indefinitely). Deduce an approximation of e correct to as many decimal places as possible.

Exercise 3. Let

$$f : \begin{array}{rcl} [-1,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \arcsin(x)\arccos(x). \end{array}$$

1. Explain why f is differentiable on $(-1,1)$, and determine an expression of f' on $(-1,1)$.
2. Determine the variations of f.

Exercise 4. The two questions of this exercise are independent.

1. Compute the derivative of the following function:

$$f : \begin{array}[t]{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \left(1+x^2\right)^x. \end{array}$$

2. Compute the derivative of the following function (simplify your answer as much as possible):

$$g : \begin{array}[t]{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \arctan\bigl(\sinh(x)\bigr) - 2\arctan\left(\mathrm{e}^x\right), \end{array}$$

   and deduce a relation between $\arctan \circ \sinh$ and $\arctan \circ \exp$.

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions such that

- $f(0) = 0$,
- f is differentiable at 0 and $f'(0) = 0$,
- g is bounded.

Show that fg is differentiable at 0 and determine the value of $(fg)'(0)$.

Exercise 6. You're given that the following function is an increasing bijection:

$$f : \begin{array}[t]{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x^3}{3} - x^2 + x. \end{array}$$

Determine the maximal subset D of $\mathbb{R}$ on which f^{-1} is differentiable, and for $x \in f^{[-1]}(D)$, give an expression of $(f^{-1})'\bigl(f(x)\bigr)$.

Exercise 7. Use equivalents to compute the value of the following limit:

$$\lim_{x\to 0} \frac{\operatorname{arcsinh}(x)^2 \ \arctan(x) \ \bigl(\cos(x)-1\bigr)}{\bigl((1+\sin(x))^{1/3}-1\bigr) \ \bigl(\cosh(2x)-1\bigr)^2}.$$

You may denote by $A(x)$ the expression that appears in the limit.

Exercise 8. Let $a \in \overline{\mathbb{R}}$ and let f and g be two functions defined in a punctured neighborhood V of a, such that $f \underset{a}{=} o(g)$ and such that g never takes the value 0. Show that $f + g \underset{a}{\sim} g$.