INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #1

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Romaric Pujol, romaric.pujol@insa-lyon.fr

Exercise 1.

- Initial step: for $n = 1$,
$$\sum_{k=1}^{1}(k+1)2^k = 2 \times 2 = 4 = 1 \times 2^{1+1},$$
hence the property is true for $n = 1$.
- Assume that there exists $n \in \mathbb{N}^*$ such that
$$(P_n) \qquad \sum_{k=1}^{n}(k+1)2^k = n2^{n+1}.$$
Then:
$$\begin{aligned}\sum_{k=1}^{n+1}(k+1)2^k &= \left(\sum_{k=1}^{n}(k+1)2^k\right) + (n+2)2^{n+1} \\ &= n2^{n+1} \ + \ (n+2)2^{n+1} && \textit{since } (P_n) \textit{ is true} \\ &= (2n+2)2^{n+1} \\ &= (n+1)2^{n+2},\end{aligned}$$
hence (P_{n+1}) is true.

Exercise 2. Let $x \in \mathbb{R}$. Then:

$$\begin{aligned}\cos(x) + \sin(x) = \sqrt{2} &\iff \frac{1}{\sqrt{2}}\cos(x) + \frac{1}{\sqrt{2}}\sin(x) = 1 \\ &\iff \sin(\pi/4)\cos(x) + \cos(\pi/4)\sin(x) = 1 \\ &\iff \sin(x + \pi/4) = 1 \\ &\iff \exists k \in \mathbb{Z},\ x + \frac{\pi}{4} = \frac{\pi}{2} + 2k\pi \\ &\iff \exists k \in \mathbb{Z},\ x = \frac{\pi}{4} + 2k\pi\end{aligned}$$

Exercise 3.

1.
$$S_0 = \sum_{k=0}^{N} 1 = N + 1$$
and
$$S_1 = \sum_{k=0}^{N} k = \frac{N(N+1)}{2}.$$

2. a) We recognize that T_2 is a telescopic sum:
$$T_2 = \sum_{k=0}^{N} x_{k+1} - x_k$$
with $x_k = k^3$, hence
$$T_2 = x_{N+1} - x_0 = (N+1)^3.$$
Also, for $k \in \mathbb{N}$,
$$(k+1)^3 - k^3 = k^3 + 3k^3 + 3k + 1 - k^3 = 3k^3 + 3k + 1,$$
hence
$$T_2 = \sum_{k=0}^{N}\big(3k^2 + 3k + 1\big) = 3\sum_{k=0}^{N} k^2 + 3\sum_{k=0}^{N} k + \sum_{k=0}^{N} 1 = 3S_2 + 3S_1 + S_0.$$

b) Hence

$$S_2 = \frac{1}{3}(N+1)^3 - S1 - \frac{1}{3}S_0 = \frac{N(N+1)(2N+1)}{6}.$$

3. We define:

$$T_3 = \sum_{k=0}^{N}(k+1)^4 - k^4$$

so that

$$T_3 = (N+1)^4 = 4S_3 + 6S_2 + 4S_1 + S_0$$

and we obtain

$$S_3 = \frac{(N+1)^4}{4} - \frac{3}{2}S_2 - S_1 - \frac{1}{4}S_0 \qquad = \cdots = \left(\frac{N(N+1)}{2}\right)^2$$

Exercise 4.

1. See Figure 1

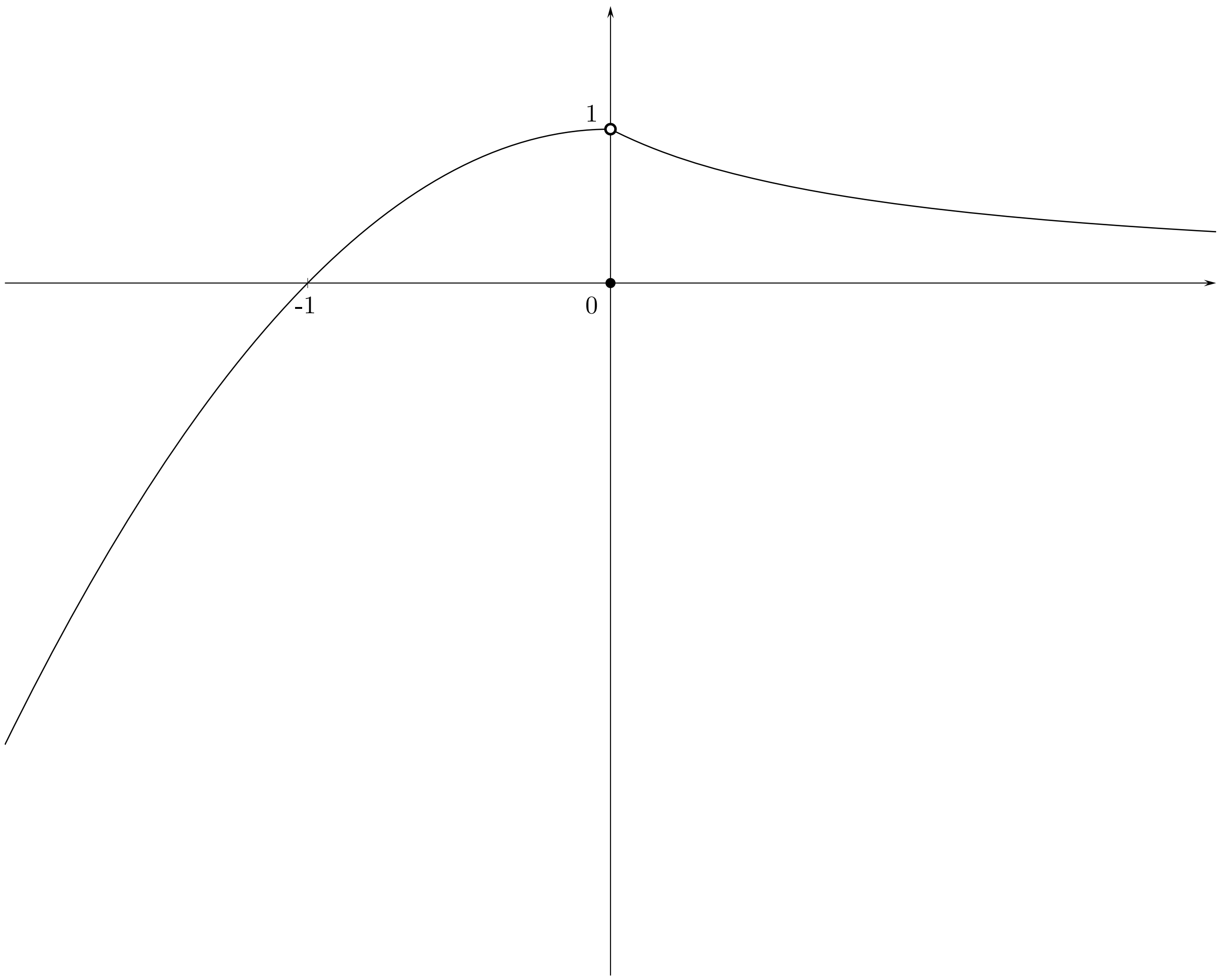

Figure 1 – Graph of function x of Exercise 4.

2. $x(\mathbb{R}) = (-\infty, 1)$.
3. x is not injective since: $x(0) = 0 = x(-1)$ and $0 \neq -1$.
   x is not surjective since 1 is in the codomain of x, but not in its range.
   x is not bijective since x is not injective.

4. x is bounded from above (and 1 is an upper bound)

5. x is not bounded from below (it's not possible to find a horizontal line such that the graph of x lies above this horizontal line).

6.
$$\begin{aligned} x(\mathbb{R}_+) &= [0,1), x(\mathbb{R}_-) &= [-\infty,1), x(\mathbb{R}_+^*) &= (0,1), x\big((-1,1]\big) &= [0,1),\\ x^{[-1]}(\mathbb{R}_-) &= (-\infty,-1]\cup\{0\}, x^{[-1]}(\mathbb{R}_-^*) &= (-\infty,-1], x^{[-1]}\big([0,1]\big) &= [-1,+\infty), x^{[-1]}\big((0,1]\big) &= (-1,0)\cup(0,+\infty), \end{aligned}$$

7. Let $a,b\in\mathbb{R}_-^*$ such that $a<b$. Observe that $a<b<0$, hence $a^2>b^2$, hence $-a^2<-b^2$, hence $1-a^2<1-b^2$, hence $x(a)<x(b)$.

   We conclude that x is increasing on $\mathbb{R}_-^*$.

Exercise 5.

1. Assume that h is injective, and let's prove that f is injective: let $x,y\in A$ such that $f(x)=f(y)$. Then $g\big(f(x)\big)=g\big(f(y)\big)$, i.e., $h(x)=h(y)$. Since h is injective, we conclude that $x=y$. Hence f is injective.

2. Assume that h is surjective, and let's prove that g is surjective: let $z\in C$. Since h is surjective, there exists $x\in A$ such that $h(x)=z$. Define $y=f(x)$. Then: $g(y)=g\big(f(x)\big)=h(x)=z$.

3. No! let $A=\{0\}$, $B=C=\{0,1\}$ and define:
$$\begin{aligned} f\ :\ A &\longrightarrow B\\ x &\longmapsto x \end{aligned}$$
   and
$$\begin{aligned} g\ :\ B &\longrightarrow C\\ x &\longmapsto x \end{aligned}$$
   Notice that f and g are well-defined, that f is injective and that g is surjective. Then:
$$\begin{aligned} h\ :\ A &\longrightarrow C\\ x &\longmapsto x \end{aligned}$$
   is not surjective.

Exercise 6. We perform the long division:

$$\begin{array}{r|l} & 2x^2-4x+10 \\ \hline x^2+1 & 2x^4-4x^3+12x^2-4x+10 \\ & -\big(2x^4 \qquad\ \ + \ 2x^2\big) \\ \cline{2-2} & \qquad -4x^3+10x^2-4x+10 \\ & \quad -\big(-4x^3 \qquad\quad -4x \qquad\big) \\ \cline{2-2} & \qquad\qquad 10x^2 \qquad +10 \\ & \qquad\quad -\big(10x^2 \qquad +10\ \big) \\ \cline{2-2} & \qquad\qquad\qquad\qquad\qquad 0 \end{array}$$

Hence
$$\forall x\in\mathbb{C},\ p(x)=\big(x^2+1\big)\big(2x^2-4x+10\big),$$
and since $i^2=-1$ we conclude that $p(i)=0$.
We now look at the polynomial X^2-2X+5: its roots are $1+2i$ and $1-2i$, so we conclude that the factorizations of p in $\mathbb{R}$ and $\mathbb{C}$ are:
$$\begin{aligned} \forall x\in\mathbb{C},\ p(x) &= 2\big(x^2+1\big)\big(x^2-2x+5\big) && \textit{in } \mathbb{R}\\ &= 2(x-i)(x+i)(x-1-2i)(x-1+2i) && \textit{in } \mathbb{C} \end{aligned}$$