SCAN 1 — Solution of Math Test #2

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Exercise 1. Yes! define the following function:

$$f\ :\ \begin{array}{rcl}\mathbb{R} & \longrightarrow & \mathbb{R}\\ x & \longmapsto & P(x)-Q(x).\end{array}$$

Notice that f is a polynomial function (since f is the sum of two polynomial functions).
We now show that every element of $\mathbb{R}_+^*$ is a root of f: let $a\in\mathbb{R}_+^*$, and define $x=\ln(a)$. Then:

$$f(a)=f\big(\mathrm{e}^x\big)=P\big(\mathrm{e}^x\big)-Q\big(\mathrm{e}^x\big)=0.$$

Hence every element of $\mathbb{R}_+^*$ is a root of f. Hence f has an infinite number of roots hence, since f is a polynomial function, we conclude that f is the nil polynomial, hence $P=Q$.

Exercise 2.

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in\mathbb{R},\ \big(0<|x+2|<\delta\implies\big|f(x)-3\big|<\varepsilon\big).$$

Exercise 3.

1. Let $x\in\mathbb{R}$. We factor $P(x)$ and $Q(x)$ by $(x-2)$, and we obtain:

$$P(x)=(x-2)(x-1)^2,\qquad\qquad Q(x)=2(x-2)(x+1).$$

   Let $x\in(-1,2)\cup(2,+\infty)$ (which is a punctured neighborhood of 2). Then:

$$\frac{P(x)}{Q(x)}=\frac{(x-2)(x-1)^2}{2(x-2)(x+1)}=\frac{(x-1)^2}{2(x+1)}\underset{x\to2}{\longrightarrow}\frac{1}{6}.$$

   Hence $\ell=1/6$.

2. Let $x\in(2,+\infty)$ (which is a punctured neighborhood of 2). Then:

$$\begin{aligned}\frac{P(x)}{Q(x)}&=\frac{x^3\left(1-\dfrac{4}{x}+\dfrac{5}{x^2}-\dfrac{2}{x^3}\right)}{x^2\left(2-\dfrac{2}{x}-\dfrac{4}{x^2}\right)}\\&=x\frac{1-\dfrac{4}{x}+\dfrac{5}{x^2}-\dfrac{2}{x^3}}{2-\dfrac{2}{x}-\dfrac{4}{x^2}}\\&\underset{x\to+\infty}{\longrightarrow}+\infty\times\frac{1}{2}&&\textit{(elementary operations on limits)}\\&=+\infty.\end{aligned}$$

   Hence $\ell'=+\infty$.

Exercise 4.

1. We use the Squeeze Theorem: by the elementary operations on limits,

$$\lim_{x\to0}1+x^2=\lim_{x\to0}1+2x^2=1,$$

   hence $\lim\limits_{x\to0}f(x)$ exists in $\mathbb{R}$ and its value is 1.

2. We use the version of the Squeeze Theorem for limits equal to $+\infty$: by the elementary operations on limits,

$$\lim_{x\to+\infty}1+x^2=+\infty$$

   hence $\lim\limits_{x\to+\infty}f(x)$ exists in $\overline{\mathbb{R}}$ and its value is $+\infty$.

3. The function sin is bounded, and by the elementary operations on limits (and the previous question):

$$\lim_{x\to+\infty}\frac{1}{f(x)}=\frac{1}{+\infty}=0.$$

Hence (by a corollary of the Squeeze Theorem), ℓ exists and

$$\ell=\lim_{x\to+\infty}\sin(x)\frac{1}{f(x)}=0.$$

Exercise 5. Let $x\in\mathbb{R}$. Then:

$$\frac{\sinh^2(2x)}{\cosh^2(x)}+4\geq 4\geq 0,$$

hence (the square root is defined and):

$$\frac{1}{2}\sqrt{\frac{\sinh^2(2x)}{\cosh^2(x)}+4}\geq 1.$$

Since the domain of cosh is $[1,+\infty)$, we conclude that A is well-defined.
By the double angle formula for sinh:

$$\sinh(2x)=2\sinh(x)\cosh(x),$$

hence

$$\sinh^2(2x)=4\sinh^2(x)\cosh^2(x),$$

hence

$$\frac{\sinh^2(2x)}{\cosh^2(x)}=4\sinh^2(x),$$

hence

$$\frac{\sinh^2(2x)}{\cosh^2(x)}+4=4\big(\sinh^2(x)+1\big)=4\cosh^2(x),$$

hence, since $\cosh(x)\geq 0$,

$$\sqrt{\frac{\sinh^2(2x)}{\cosh^2(x)}+4}=2\cosh(x),$$

hence

$$\frac{1}{2}\sqrt{\frac{\sinh^2(2x)}{\cosh^2(x)}+4}=\cosh(x)=\cosh\big(|x|\big).$$

(The last equality follows form the fact that cosh is even). Hence

$$A=\operatorname{arccosh}\Big(\cosh\big(|x|\big)\Big)=|x|,$$

since arccosh is the inverse function of the restriction of cosh to $\mathbb{R}_+$ (and $|x|\in\mathbb{R}_+$).

Exercise 6. Let $x\in\mathbb{R}_+^*$ (which is a right-sided punctured neighborhood of 0). Then

$$x^x=\mathrm{e}^{x\ln(x)}.$$

We know that:

$$\lim_{x\to 0^+}x\ln(x)=0,$$

hence, by the Composition of Limits Theorem

$$\ell=\lim_{x\to 0^+}\mathrm{e}^{x\ln(x)}=\lim_{y\to 0}\mathrm{e}^y=1.$$

Exercise 7.

1. See Figure 2

2.

$$A=\sup f=1,\qquad B=\max f\text{ DNE},\qquad C=\inf f=0,\qquad D=\min f=0.$$

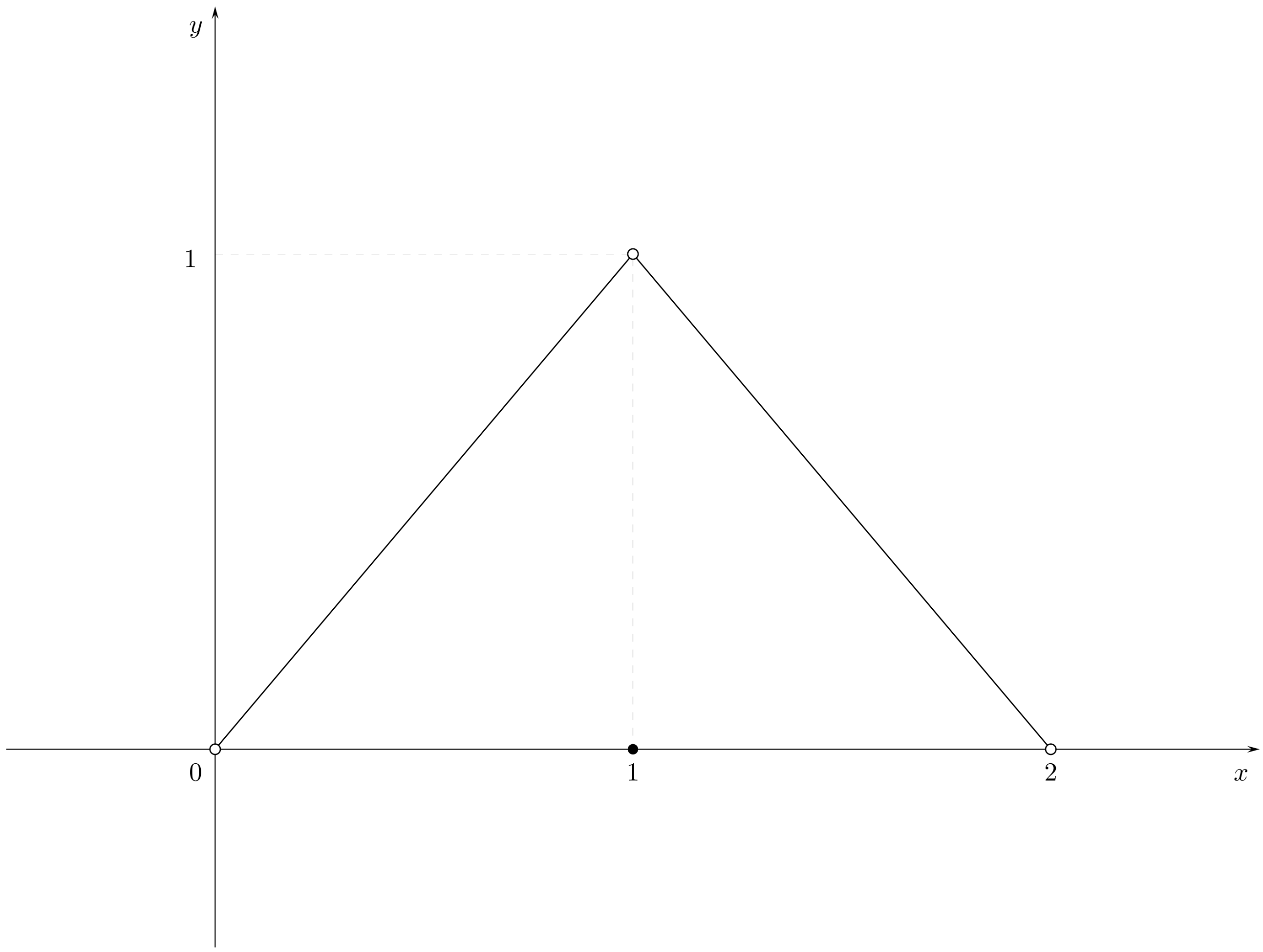

Figure 2 – Graph of the function f of Exercise 7