INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #2 — 1h30 *November 27, 2020*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let P and Q be two polynomial functions such that

$$\forall x \in \mathbb{R},\ P\big(\mathrm{e}^x\big) = Q\big(\mathrm{e}^x\big).$$

Can we deduce that $P = Q$? Justify your answer.

Exercise 2. Let f be a function defined in a punctured neighborhood of -2. Write the "ε-δ" definition of the following limit:

$$\lim_{x\to -2} f(x) = 3.$$

Exercise 3. Let P and Q be the polynomial functions defined by:

$$P : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 - 4x^2 + 5x - 2 \end{array} \qquad \text{and} \qquad Q : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 2x^2 - 2x - 4. \end{array}$$

1. After noticing that 2 are roots of P and Q, determine whether the limit

$$\ell = \lim_{x\to 2} \frac{P(x)}{Q(x)}$$

exists (in $\overline{\mathbb{R}}$). If the limit exists, determine its value.

2. Determine the value of the following limit (in $\overline{\mathbb{R}}$):

$$\ell' = \lim_{x\to +\infty} \frac{P(x)}{Q(x)}.$$

Justify your answer as concisely as possible.

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that:

$$\forall x \in \mathbb{R},\ 1 + x^2 \leq f(x) \leq 1 + 2x^2.$$

1. Show that $\lim\limits_{x\to 0} f(x)$ exists in $\mathbb{R}$ and determine its value.

2. Show that $\lim\limits_{x\to +\infty} f(x)$ exists in $\overline{\mathbb{R}}$ and determine its value.

3. Does the following limit exist in $\overline{\mathbb{R}}$?

$$\ell = \lim_{x\to +\infty} \frac{\sin(x)}{f(x)}.$$

If ℓ exists, determine its value.

Exercise 5. Let $x \in \mathbb{R}$. Show that following expression is well-defined, and simplify it as much as you can:

$$A = \operatorname{arccosh}\left(\frac{1}{2}\sqrt{\frac{\sinh^2(2x)}{\cosh^2(x)} + 4}\right).$$

Exercise 6. Show that the following limit exists in $\mathbb{R}$ and determine its value.

$$\ell = \lim_{x \to 0^+} x^x.$$

Exercise 7. Let f be the function defined as:

$$\begin{array}{rccl} f : & (0,2) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} x & \text{if } x \in (0,1) \\ 0 & \text{if } x = 1 \\ 2 - x & \text{if } x \in (1,2). \end{cases} \end{array}$$

1. Sketch the graph of f.
2. Determine the following values (no justifications required). If a value doesn't exist, write "DNE".

$$A = \sup f, \qquad B = \max f, \qquad C = \inf f, \qquad D = \min f.$$