INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #3

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Exercise 1.

1. Let $n \in \mathbb{N}^*$.

$$u_{n+1} - u_n = \frac{1}{(n+1)!} > 0,$$

hence $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

$$\begin{aligned} v_{n+1} - v_n &= u_{n+1} - u_n + \frac{1}{(n+1)\cdot(n+1)!} - \frac{1}{n\cdot n!} \\ &= \frac{1}{(n+1)!} + \frac{1}{(n+1)\cdot(n+1)!} - \frac{1}{n\cdot n!} \\ &= \frac{n(n+1) + n - (n+1)^2}{n(n+1)(n+1)!} \\ &= \frac{n^2 + 2n - (n+1)^2}{n(n+1)(n+1)!} \\ &= -\frac{1}{n(n+1)(n+1)!} < 0, \end{aligned}$$

hence $(v_n)_{n\in\mathbb{N}^*}$ is decreasing.

2. For $n \in \mathbb{N}^*$:

$$u_n - v_n = \frac{1}{n\cdot n!} \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent sequences, hence they converge to the same limit.

3. a) Since the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing and its limit is e and since the sequence $(v_n)_{n\in\mathbb{N}^*}$ is decreasing and its limit is also e we conclude that:

$$\forall n \in \mathbb{N}^*,\ u_n < \mathrm{e} < v_n,$$

and in particular (taking $n = q$):

$$u_q < \mathrm{e} < v_q,$$

hence

$$u_q \cdot q! < \mathrm{e}\cdot q! < v_q \cdot q! = u_q \cdot q! + \frac{1}{q}.$$

b)

$$q! \cdot u_q = \sum_{k=0}^{q} \frac{q!}{k!}.$$

All the terms in this sum are non-negative integers: indeed,

- if $k = q$ then $q!/k! = 1 \in \mathbb{N}$,
- and if $0 \le k \le q-1$,

$$\frac{q!}{k!} = \prod_{m=k+1}^{q} m,$$

which is a product of non-negative integers, hence a non-negative integer.

Hence $q! \cdot u_q \in \mathbb{N}$.

c) Since $q \ge 1$, $1/q \le 1$ and hence:

$$u_q \cdot q! < \mathrm{e}\cdot q! < u_q \cdot q! + 1,$$

hence

$$0 < \mathrm{e}\cdot q! - u_q \cdot q! < 1.$$

Now observe that $\mathrm{e}\cdot q! = p\cdot(q-1)! \in \mathbb{N}$, hence $\mathrm{e}\cdot q! - u_q\cdot q! \in \mathbb{N}$, which appears to be a contradiction, since there are no integers in $(0,1)$

Exercise 2.

1. We apply the Intermediate Value Theorem to the function f_n on $[0,1]$:
   - The function f_n is continuous (it's a polynomial function),
   - $[0,1]$ is an interval,
   - $f_n(0) = -1 < 0$,
   - $f_n(1) = 2 > 0$,

   hence, by the Intermediate Value Theorem, there exists $x_n \in (0,1)$ such that $f_n(x_n) = 0$.

   To show that such an x_n is unique, we only need to show that the function f_n is injective: we know that for $\alpha > 0$, the function $x \mapsto x^\alpha$ is increasing on $\mathbb{R}_+$, hence f_n is a sum of increasing (and constant) functions, hence f_n is increasing. Hence f_n is injective, and x_n is indeed unique.

2. a) Let $n \in \mathbb{N}$. Then:
   $$u_{n+1} - u_n = f_{n+1}(a) - f_n(a) = a^{n+1} - a^n = a^n(a-1) < 0.$$
   Hence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

   b) Let $n \in \mathbb{N}$: from the previous question (with $a = x_{n+1}$) we know that
   $$f_n(x_{n+1}) > f_{n+1}(x_{n+1}) = 0 = f_n(x_n).$$
   Since f_n is increasing, we conclude that $x_{n+1} > x_n$,for otherwise, if $x_{n+1} \leq x_n$, we would have $f_n(x_{n+1}) \leq f_n(x_n)$, which is impossible. Hence $(x_n)_{n\in\mathbb{N}}$ is increasing.

   c) Since the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded from above (1 is an upper bound) and increasing we know, by the Monotone Limit Theorem, that $(x_n)_{n\in\mathbb{N}}$ converges.

3. Let $n \in \mathbb{N}$. Then:
   $$f_n\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^n + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right) - 1 = \left(\frac{3}{4}\right)^n + \frac{5}{16} \geq \frac{5}{16} > 0.$$
   Since the function f_n is increasing, we can't have $x_n \geq 3/4$ for otherwise we would have $f_n(x_n) \geq f(3/4) > 0$.

   From:
   $$0 < x_n < \frac{3}{4}$$
   we deduce:
   $$0 < x_n^n < \left(\frac{3}{4}\right)^n.$$
   Since $3/4 \in (0,1)$,
   $$\lim_{n\to+\infty} \left(\frac{3}{4}\right)^n = 0$$
   and by the Squeeze Theorem we deduce that $\lim_{n\to+\infty} x_n^n = 0$.

4. We know that:
   $$\forall n \in \mathbb{N},\ x_n^n + x_n^2 + x_n - 1 = 0,$$
   hence, by the elementary operations on limits (and using the result of the previous question and the fact that $(x_n)_{n\in\mathbb{N}}$ converges to ℓ):
   $$\ell^2 + \ell - 1 = 0.$$
   The solutions of this quadratic are:
   $$\frac{-1 \pm \sqrt{5}}{2}.$$
   Since the values of $(x_n)_{n\in\mathbb{N}}$ are in $(0,1)$, we deduce that $\ell \in [0,1]$, and the only possibility is:
   $$\ell = \frac{-1 + \sqrt{5}}{2}.$$

Exercise 3. Covered in class…

Exercise 4.

1. Domain: $[-1, 1]$. Range: $[0, \pi]$.

2. Let $x \in \mathbb{R}$. Then:

$$-1 \leq \frac{1-x^2}{1+x^2} \leq 1 \iff -1-x^2 \leq 1-x^2 \leq 1+x^2 \iff -1 \leq 1 \leq 1+x^2,$$

which is always true (as $x^2 \geq 0$). Hence $D = \mathbb{R}$.

3. a)

$$\begin{aligned} A(x) &= \frac{1}{2}\big(1-\cos\big(f(x)\big)\big) \\ &= \frac{1}{2}\left(1-\cos\left(\arccos\left(\frac{1-x^2}{1+x^2}\right)\right)\right) \\ &= \frac{1}{2}\left(1-\frac{1-x^2}{1+x^2}\right) \\ &= \frac{x^2}{1+x^2}, \end{aligned}$$

and

$$\begin{aligned} B(x) &= \frac{1}{2}\big(1+\cos\big(f(x)\big)\big) \\ &= \frac{1}{2}\left(1+\cos\left(\arccos\left(\frac{1-x^2}{1+x^2}\right)\right)\right) \\ &= \frac{1}{2}\left(1+\frac{1-x^2}{1+x^2}\right) \\ &= \frac{1}{1+x^2}. \end{aligned}$$

b) We notice that $B(x) \neq 0$, hence $\cos\big(f(x)/2\big) \neq 0$, hence $C(x) = \tan\big(f(x)/2\big)$ is well-defined. Moreover:

$$\tan^2\left(\frac{f(x)}{2}\right) = \frac{A(x)}{B(x)} = x^2,$$

and we conclude:

$$C(x) = \tan\left(\frac{f(x)}{2}\right) = |x|.$$

c) Since the range of arccos is $[0, \pi]$, we conclude that the range of f is a subset of $[0, \pi]$, hence $f(x)/2 \in [0, \pi/2]$. Since $B(x) \neq 0$ we moreover conclude that $f(x)/2 \in [0, \pi/2) \subset (-\pi/2, \pi/2)$. Hence:

$$f(x)/2 = \arctan\big(C(x)\big) = \arctan|x|,$$

hence:

$$f(x) = 2\arctan|x| = 2\big|\arctan(x)\big|.$$

Exercise 5. Notice that

$$g(0) = \sqrt{0} + 0f(0) = 0.$$

Let $x \in \mathbb{R}_+^*$. Then:

$$\frac{g(x)-g(0)}{x-0} = \frac{\sqrt{x}+xf(x)}{x} = \frac{1}{\sqrt{x}} + f(x).$$

Now,

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \lim_{x\to 0^+} x^{-1/2} = +\infty,$$

and since f is bounded we conclude, by (a corollary of) the Squeeze Theorem, that

$$\lim_{x\to 0^+} \frac{g(x)-g(0)}{x-0} = +\infty,$$

hence g is not differentiable (from the right) at 0.