INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #4

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Exercise 1.

1. If f is of class C^N on $[a,b]$ and $N+1$ times differentiable on (a,b), there exists $c\in(a,b)$ such that:

$$f(b)=\sum_{k=0}^{N}\frac{f^{(k)}(a)}{k!}(b-a)^k \;+\; \frac{f^{(N+1)}(c)}{(N+1)!}(b-a)^{N+1}.$$

2. The function exp is of class C^4 on $[0,1/2]$ and four times differentiable on $(0,1/2)$ hence by the Taylor–Lagrange formula, there exists $c\in(0,1/2)$ such that

$$\exp(1/2)=\sqrt{\mathrm{e}}=1+\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{6}\left(\frac{1}{2}\right)^3+\frac{1}{24}\left(\frac{1}{2}\right)^4+\frac{\mathrm{e}^c}{5!}\left(\frac{1}{2}\right)^5.$$

Now,

$$\begin{aligned}1+\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{6}\left(\frac{1}{2}\right)^3+\frac{1}{24}\left(\frac{1}{2}\right)^4&=1+\frac{1}{2}+\frac{1}{8}+\frac{1}{6\times 8}+\frac{1}{24\times 16}\\&=\frac{384+192+48+8+1}{384}=\frac{633}{384}=\frac{211}{128}\end{aligned}$$

and since exp is increasing and $0<c<1/2$,

$$0<\mathrm{e}^c<\sqrt{e}<2.$$

Hence

$$0<\frac{\mathrm{e}^c}{5!}\left(\frac{1}{2}\right)^5<\frac{1}{16\times 5!}=\frac{1}{16\times 120}=\frac{1}{1920}<\frac{1}{2000}=0.0005.$$

3. From the value given, we deduce:

$$1.6484<\frac{211}{128}<1.6485$$

and hence:

$$1.6484<\sqrt{\mathrm{e}}=\mathrm{e}^{1/2}<1.6485+0.0005=1.649.$$

Since the right inequality is a strict inequality, we conclude that

$$\sqrt{\mathrm{e}}=1.648\ldots$$

Exercise 2.

1. The Mean Value Theorem is: let $a,b\in\mathbb{R}$ with $a\neq b$, and let $f:[a,b]\to\mathbb{R}$ such that f is continuous on $[a,b]$ and f is differentiable on (a,b). Then there exists $c\in(a,b)$ such that $f(b)-f(a)=f'(c)(b-a)$.

   It is seen as a special case of Cauchy's Mean Value Theorem by taking $g(x)=x$, since in this case $g'(x)=1$.

2. Since f and g are continuous on $[a,b]$ and differentiable on (a,b), by elementary operations, h is also continuous on $[a,b]$ and differentiable on (a,b). Moreover,

$$h(a)=\big(g(b)-g(a)\big)f(a)-\big(f(b)-f(a)\big)g(a)=g(b)f(a)-f(b)g(a)$$

and

$$h(b)=\big(g(b)-g(a)\big)f(b)-\big(f(b)-f(a)\big)g(b)=-g(a)f(b)+f(a)g(b),$$

and we conclude that $h(a)=h(b)$. Hence we can apply Rolle's Theorem, and we conclude that there exists $c\in(a,b)$ such that $h'(c)=0$. The result follows from the fact that

$$h'(c)=\big(g(b)-g(a)\big)f'(c)-\big(f(b)-f(a)\big)g'(c).$$

3. All the conditions of Cauchy's Mean Value Theorem are fulfilled for the functions x and y on $[0,1]$. Hence there exists $t_0 \in (0,1)$ such that

$$\big(x(1)-x(0)\big)y'(t_0) = \big(y(1)-y(0)\big)x'(t_0).$$

The velocity of the particle at t_0 is $\vec{v}_{t_0} = \big(x'(t_0), y'(t_0)\big)$ and the vector $\overrightarrow{M_0M_1}$ is $\big(x(1)-x(0), y(1)-y(0)\big)$. By using the cross-product property to check whether these vectors are collinear, we find:

$$\big(x(1)-x(0)\big)y'(t_0) - \big(y(1)-y(0)\big)x'(t_0) = 0$$

i.e., $\overrightarrow{M_0M_1}$ and $\vec{v}_{t_0}$ are collinear. See Figure 3.

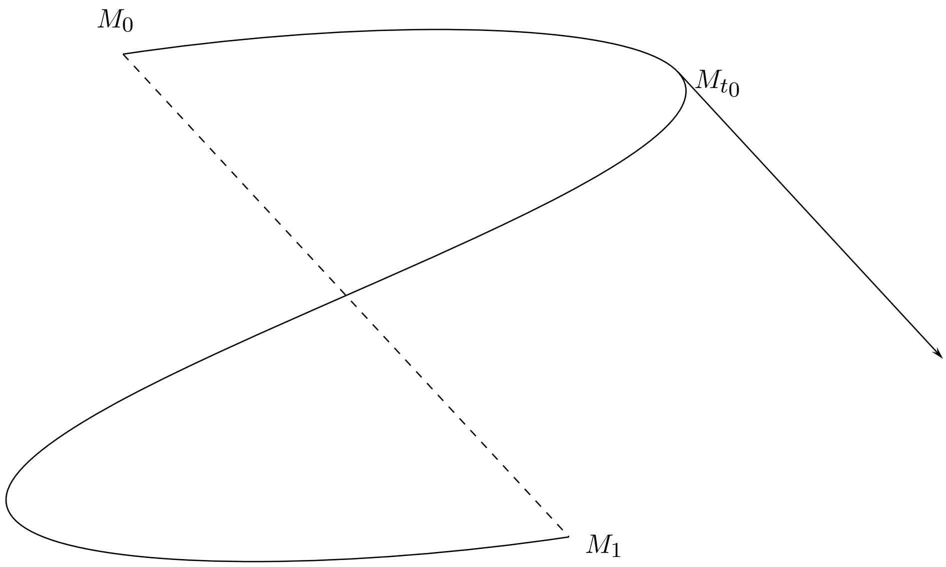

Figure 3 – Trajectory of a particle: there exists a point where its velocity is collinear to $\overrightarrow{M_0M_1}$.

Exercise 3.

1. For $x \in \mathbb{R}_+^*$,

$$\ln\big(e^x + x\big) = \ln\big(e^x(xe^{-x}+1)\big) = x + \ln(xe^{-x}+1).$$

Now,

$$\frac{\ln(xe^{-x}+1)}{x} \underset{x\to+\infty}{\longrightarrow} 0,$$

hence $\ln(xe^{-x}+1) \underset{x\to+\infty}{=} o(x)$, and we conclude that

$$\ln\big(e^x + x\big) \underset{x\to+\infty}{\sim} x.$$

2. Let $x \in \mathbb{R}^*$. Then:
$$\big(\cosh(x)\big)^{1/x^2} = \exp\left(\frac{1}{x^2}\ln\big(\cosh(x)\big)\right).$$
Now we know that $\cosh(x) \underset{x\to 0}{\longrightarrow} 1$ and that $\ln(X) \underset{X\to 1}{\sim} X-1$, hence
$$\ln\big(\cosh(x)\big) \underset{x\to 0}{\sim} \cosh(x)-1 \underset{x\to 0}{\sim} \frac{x^2}{2}.$$
Finally,
$$\frac{1}{x^2}\ln\big(\cosh(x)\big) \underset{x\to 0}{\sim} \frac{1}{2} \underset{x\to 0}{\longrightarrow} \frac{1}{2},$$
and we conclude, by composition of limits, that
$$\lim_{x\to 0}\big(\cosh(x)\big)^{1/x^2} = \mathrm{e}^{1/2}.$$

3. Let $x \in (-1,+\infty)$. Then:
$$\sqrt{1+x^2}-\cos(x) = \sqrt{1+x^2}-1+1-\cos(x).$$
We know that
$$\sqrt{1+x^2}-1 \underset{x\to 0}{\sim} \frac{x^2}{2} \qquad \text{and} \qquad 1-\cos(x) \underset{x\to 0}{\sim} \frac{x^2}{2}.$$
Hence:
$$\frac{\sqrt{1+x^2}-\cos(x)}{x^2} = \frac{\sqrt{1+x^2}-1}{x^2} + \frac{1-\cos(x)}{x^2} \underset{x\to 0}{\longrightarrow} \frac{1}{2}+\frac{1}{2} = 1,$$
hence
$$\sqrt{1+x^2}-\cos(x) \underset{x\to 0}{\sim} x^2.$$

Exercise 4.

1. By induction: the property is true for $n=0$. Assume it is true for some $n \in \mathbb{N}$, then $f_{n+1}(0) = \mathrm{e}^{f_n(0)}-1 = \mathrm{e}^0-1 = 0$.

2. By the Taylor–Young theorem, we only need to show that all the f_n's are twice differentiable at 0: we can proceed by induction: f_0 is clearly twice differentiable at 0. Assume that f_n is twice differentiable at 0 for some $n \in \mathbb{N}$ then, since f_{n+1} is obtained by composition the twice differentiable function exp and f_n we conclude, by the Chain Rule, that f_{n+1} is also twice differentiable at 0.

3. We know that $\mathrm{e}^X-1 \underset{X\to 0}{=} X+X^2/2+o\big(X^2\big)$. Hence, by the substitution $X = f_n(x) \underset{x\to 0}{\longrightarrow} 0$:
$$\begin{aligned} f_{n+1}(x) &\underset{x\to 0}{=} f_n(x)+f_n(x)^2/2+o\big(f_n(x)^2\big) \\ &\underset{x\to 0}{=} \alpha_n x+\beta_n x^2+\frac{1}{2}\alpha_n^2x^2+o\big(x^2\big)+o\big(f_n(x)^2\big). \end{aligned}$$
Now $o\big(f_n(x)^2\big) \underset{x\to 0}{=} o(x^2)$, hence
$$f_{n+1}(x) \underset{x\to 0}{=} \alpha_n x+\left(\beta_n+\frac{1}{2}\alpha_n^2\right)x^2+o\big(x^2\big).$$
By identification, we conclude that
$$\begin{cases} \alpha_{n+1} = \alpha_n \\ \beta_{n+1} = \beta_n+\alpha_n^2/2. \end{cases}$$

4. Since $\alpha_0 = 1$, we conclude that all the α_n's are equal to 1, and hence
$$\forall n \in \mathbb{N},\ \beta_{n+1} = \beta_n+\frac{1}{2},$$
where we recognize an arithmetic sequence. Since $\beta_0 = 0$, we conclude that:
$$\forall n \in \mathbb{N},\ \beta_n = \frac{n}{2}.$$

5. Since:
$$f_{n+1}(x) - f_n(x) \underset{x\to 0}{=} \frac{1}{2}x^2 + o\big(x^2\big)$$
and since the leading term $x^2/2 > 0$, we conclude that the graph of f_{n+1} is above that of f_n in a neighborhood of 0.

6. From the coefficients of α_n and β_n obtained in the previous question we conclude:
$$\forall n \in \mathbb{N},\ f_n'(0) = 1, \qquad f_n''(0) = n.$$

Exercise 5.

1. We use:
$$\cos(X) \underset{X\to 0}{=} 1 - \frac{X^2}{2} + o\big(X^3\big)$$
and
$$X = \ln(1+x) \underset{x\to 0}{=} x - \frac{x^2}{2} + o\big(x^2\big) \underset{x\to 0}{\sim} x \underset{x\to 0}{\longrightarrow} 0,$$
and substitute:
$$\begin{aligned}\cos\big(\ln(1+x)\big) &\underset{x\to 0}{=} 1 - \frac{1}{2}\left(x - \frac{x^2}{2} + o\big(x^2\big)\right)^2 + o\big(x^3\big)\\ &\underset{x\to 0}{=} 1 - \frac{1}{2}\big(x^2 - x^3\big) + o\big(x^3\big)\end{aligned}$$
and we conclude:
$$\cos\big(\ln(1+x)\big) - 1 + x^2/2 \underset{x\to 0}{=} \frac{x^3}{2} + o\big(x^3\big) \underset{x\to 0}{\sim} \frac{x^3}{2}.$$
Hence
$$\lim_{x\to 0} \frac{\cos\big(\ln(1+x)\big) - 1 + x^2/2}{x^3} = \frac{1}{2}.$$

2. We use:
$$\sin(x) \underset{x\to 0}{=} x - \frac{x^3}{6} + o\big(x^3\big)$$
and
$$\ln(1+x) \underset{x\to 0}{=} x - \frac{x^2}{2} + \frac{x^3}{3} + o\big(x^3\big),$$
and by a long division we obtain:
$$f(x) \underset{x\to 0}{=} 1 + \frac{x}{2} - \frac{x^2}{4} + o\big(x^2\big).$$
From here we deduce:
   - f possesses an extension by continuity at 0 since $\lim\limits_{x\to 0} f(x) = 1 \in \mathbb{R}$.
   - Moreover, since f possesses a first order Taylor–Young expansion at 0, we conclude that f is differentiable at 0, and we also read from its Taylor–Young expansion that an equation of its tangent line at 0 is:
   $$\Delta : y = 1 + \frac{x}{2}.$$
   - From the second order term (which is negative in a neighborhood of 0), we conclude that the graph of f lies below Δ in a neighborhood of 0 (see Figure 4).

Exercise 6.

1. We set $u(x) = x$ and $v'(x) = \mathrm{e}^x$ so that $u'(x) = 1$ and $v(x) = \mathrm{e}^x$. Clearly u and v are of class C^1 and hence, by integration by parts:
$$\begin{aligned}\int_0^1 x\mathrm{e}^x \,\mathrm{d}x &= \big[x\mathrm{e}^x\big]_{x=0}^{x=1} - \int_0^1 \mathrm{e}^x \,\mathrm{d}x\\ &= \mathrm{e} - \big[\mathrm{e}^x\big]_{x=0}^{x=1}\\ &= \mathrm{e} - (\mathrm{e} - 1)\\ &= 1.\end{aligned}$$

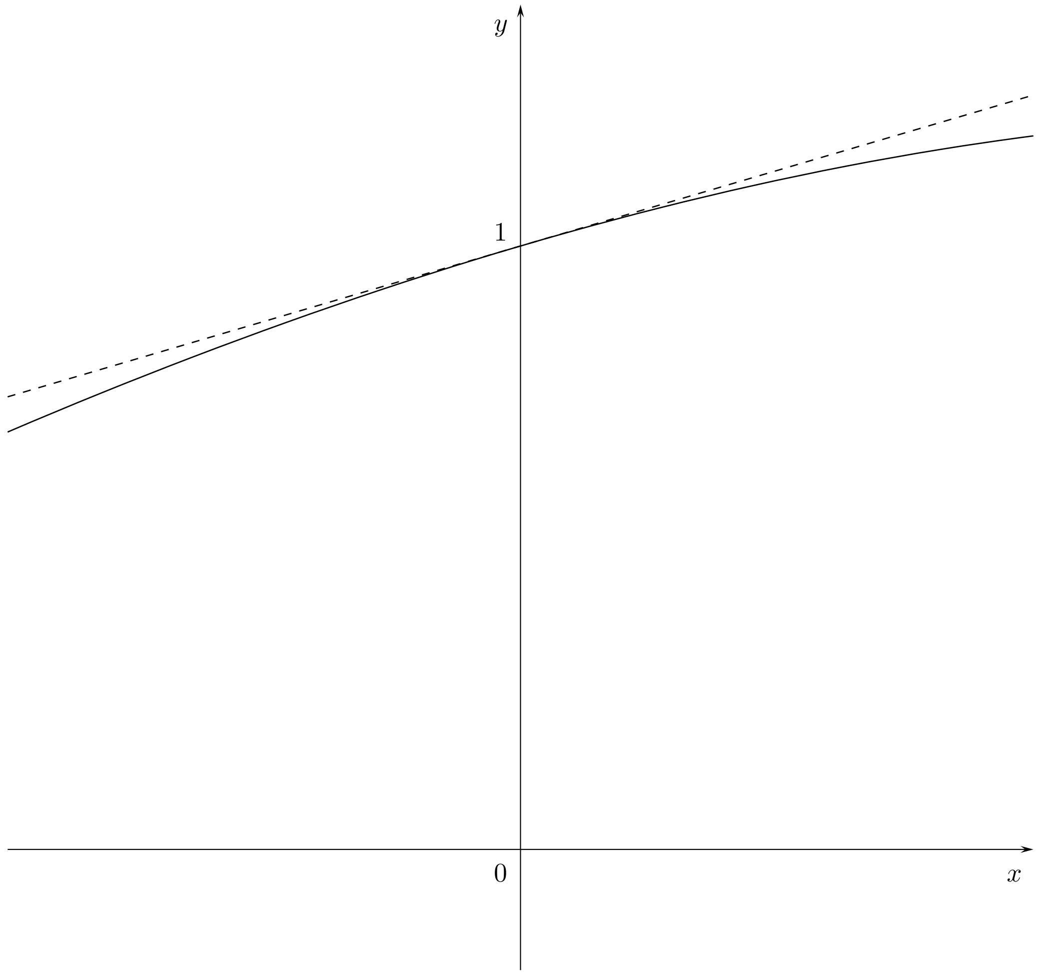

Figure 4 – Graph of function f of Exercise 5, as well as its tangent line at $(0, 1)$ (dashed)

2. a) Since $\alpha \neq 1$:

$$I(A) = \int_A^1 \frac{1}{x^\alpha}\,\mathrm{d}x = \left[-\frac{1}{(\alpha-1)x^{\alpha-1}}\right]_{x=A}^{x=1}$$
$$= -\frac{1}{\alpha-1} + \frac{1}{(\alpha-1)A^{\alpha-1}}$$

b) Hence:

$$\lim_{A\to 0^+} I(A) = \begin{cases} +\infty & \text{if } \alpha > 1 \\ \dfrac{1}{1-\alpha} & \text{if } \alpha < 1 \end{cases}$$