INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #4 — 1h30

March 26, 2021

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The goal of this exercise is to compute an approximation value of $\sqrt{e}$.

1. Recall the N-th order Taylor–Lagrange formula for a function f on an interval $[a, b]$.
2. Show, by the applying the 4-th order Taylor–Lagrange formula to the function exp, that

$$\frac{211}{128} < \sqrt{\mathrm{e}} < \frac{211}{128} + \frac{1}{1920} < \frac{211}{128} + 0.0005.$$

   You may use, without any justifications, the fact that $0 < \mathrm{e} < 4$.

3. You're given the *exact* value:

$$\frac{211}{128} = 1.6484375.$$

   Deduce an approximation of $\sqrt{\mathrm{e}}$ correct to as many decimal places as you can.

Exercise 2. The goal of this exercise is to prove the following theorem, known as Cauchy's Mean Value Theorem:

> **Theorem.** *Let $a, b \in \mathbb{R}$ with $a \neq b$ and let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be such that:*
>
> - *f and g are continuous on $[a, b]$,*
> - *f and g are differentiable on (a, b).*
>
> *Then there exists $c \in (a, b)$ such that*
>
> $$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c).$$

We assume that a, b, f, g satisfy the condition of the theorem.

1. State the (standard) Mean Value Theorem, and explain why it is a particular case of Cauchy's Mean Value Theorem.
2. *Proof of Cauchy's Mean Value Theorem:* define the function:

$$\begin{array}{rccl} h: & [a,b] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \big(g(b)-g(a)\big)f(x) - \big(f(b)-f(a)\big)g(x). \end{array}$$

   Prove Cauchy's Mean Value Theorem by applying Rolle's Theorem to h.

3. *Application:* for $t \in [0, 1]$ we denote by $M_t = \big(x(t), y(t)\big)$ the position of a particle at time t in the plane. We assume that x and y are differentiable on $[0, 1]$.
   Show that there exists $t_0 \in (0, 1)$ such that the velocity of the particle at time t_0 is collinear (i.e., parallel) to the vector $\overrightarrow{M_0M_1}$. Sketch a graph to illustrate this result.

Exercise 3. The questions of this exercise are independent from each other.

1. Find the simplest equivalent as $x \to +\infty$ of $\ln(\mathrm{e}^x + x)$.
2. Use equivalents to determine the value of the following limit $\lim_{x\to 0}\big(\cosh(x)\big)^{1/x^2}$.
3. Find the simplest equivalent as $x \to 0$ of $\sqrt{1+x^2} - \cos(x)$.

Exercise 4. We define:

$$f_0 : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto x \end{array} \qquad \text{and} \qquad \forall n \in \mathbb{N}, \quad f_{n+1} : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto e^{f_n(x)} - 1. \end{array}$$

1. Show that

$$\forall n \in \mathbb{N},\ f_n(0) = 0.$$

2. Explain why, for all $n \in \mathbb{N}$ the function f_n possesses a second order Taylor–Young expansion at 0. We denote by α_n and β_n the coefficients such that

$$f_n(x) \underset{x \to 0}{=} \alpha_n x + \beta_n x^2 + o(x^2).$$

3. For $n \in \mathbb{N}$, express α_{n+1} and β_{n+1} in terms of α_n and β_n.
4. Deduce a formula for the coefficients α_n and β_n that only depend on n.
5. For $n \in \mathbb{N}$, deduce the relative position of the graph of f_n and f_{n+1} in a neighborhood of 0.
6. For $n \in \mathbb{N}$, deduce the value of $f_n'(0)$ and $f_n''(0)$.

Exercise 5. The questions of this exercise are independent from each other.

1. Use a Taylor expansion to determine the value of the following limit:

$$\lim_{x \to 0} \frac{\cos(\ln(1+x)) - 1 + x^2/2}{x^3}.$$

2. Let

$$f : \begin{array}{l} (-1,0) \cup (0,+\infty) \longrightarrow \mathbb{R} \\ x \longmapsto \dfrac{\sin(x)}{\ln(1+x)}. \end{array}$$

Show that f possesses an extension by continuity at 0 that we still denote by f, and that f is differentiable at Determine an equation of the tangent line Δ to the graph of f at 0 as well as the relative position of f with respe to Δ in a neighborhood of 0. Sketch a graph of f in a neighborhood of 0.

Exercise 6. The questions of this exercise are independent from each other.

1. Use an integration by parts to compute the value of

$$\int_0^1 x e^x \, dx.$$

2. Let $\alpha \in (0,1) \cup (1,+\infty)$, and let $A \in \mathbb{R}_+^*$.

a) Determine the value of the following integral:

$$I(A) = \int_A^1 \frac{1}{x^\alpha} dx$$

in terms of α.

b) Deduce the value (in $\overline{\mathbb{R}}$) of the limit $\lim_{A \to 0^+} I(A)$ in terms of α.