INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #5

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Exercise 1.

1. Let $x \in \mathbb{R}_+^*$. Then $x + x^2 > 0$, hence $\left[x, x + x^2\right] \subset \mathbb{R}^*$, and the function $t \mapsto \cos(t)/t^2$ is continuous on $\left[x, x + x^2\right]$, hence $\varphi(x)$ is well defined.

2. The function
$$f : \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{\cos(t)}{t^2} \end{array}$$
is continuous on $\mathbb{R}_+^*$, hence admits an antiderivative F on $\mathbb{R}_+^*$. Then, by the Fundamental Theorem of Calculus, for $x \in \mathbb{R}_+^*$,
$$\varphi(x) = F\bigl(x + x^2\bigr) - F(x).$$
Since F is differentiable we conclude, by the Chain Rule (and Addition Rule) that φ is differentiable and that:
$$\begin{aligned} \forall x \in \mathbb{R}_+^*,\ \varphi'(x) &= (1 + 2x)F'\bigl(x + x^2\bigr) - F'(x) \\ &= (1 + 2x)f\bigl(x + x^2\bigr) - f(x) \\ &= (1 + 2x)\frac{\cos\bigl(x + x^2\bigr)}{\bigl(x + x^2\bigr)^2} - \frac{\cos x}{x^2}. \end{aligned}$$

3. Let $x \in \mathbb{R}_+^*$. The function cos is continuous on $\left[x, x+x^2\right]$, and the function $t \mapsto 1/t^2$ is (piecewise) continuous and positive on $\left[x, x + x^2\right]$ hence, by MVT2, there exists $c_x \in \left[x, x + x^2\right]$ such that
$$\begin{aligned} \varphi(x) &= \cos(c_x) \int_x^{x+x^2} \frac{\mathrm{d}t}{t^2} \\ &= \cos(c_x) \left[-\frac{1}{t}\right]_{t=x}^{t=x+x^2} \\ &= \cos(c_x) \left(-\frac{1}{x + x^2} + \frac{1}{x}\right) \\ &= \cos(c_x) \frac{-x + x + x^2}{x(x + x^2)} \\ &= \cos(c_x) \frac{1}{1 + x}. \end{aligned}$$

4. Let $x \in \mathbb{R}_+^*$.
   - Since $c_x \in [x, x + x^2]$, we conclude that $c_x \underset{x\to 0^+}{\longrightarrow} 0$, hence $\lim\limits_{x\to 0^+} \cos(c_x) = 1$, hence $\lim\limits_{x\to 0^+} \varphi(x) = 1$.
   - Moreover,
   $$\bigl|\varphi(x)\bigr| \leq \frac{1}{1 + x} \underset{x\to+\infty}{\longrightarrow} 0,$$
   hence $\lim\limits_{x\to+\infty} \varphi(x) = 0$.

Exercise 2.

1. Let $n \in \mathbb{N}$. Then
$$I_{n+1} - I_n = \int_0^1 \mathrm{e}^{\alpha t} t^n (t - 1)\,\mathrm{d}t.$$
Now,
$$\forall t \in [0, 1],\ \mathrm{e}^{\alpha t} t^n (t - 1) \leq 0$$
and the function $t \mapsto \mathrm{e}^{\alpha t} t^n (t - 1)$ is continuous and not identically nil. Hence (since the endpoints of the integral are in increasing order, i.e., $0 < 1$) we conclude that $I_{n+1} - I_n < 0$, hence the sequence $(I_n)_{n\geq 0}$ is decreasing.

Since

$$\forall t \in [0,1],\ \mathrm{e}^{\alpha t} t^n \geq 0$$

we conclude that $I_n \geq 0$, hence the sequence $(I_n)_{n\geq 0}$ is bounded from below. Hence the sequence $(I_n)_{n\geq 0}$ is convergent.

2. Let $n \in \mathbb{N}$. Then, by an integration by parts (differentiating $t \mapsto \mathrm{e}^{\alpha t}$ and antidifferentiating $t \mapsto t^n$):

$$\begin{aligned} I_n &= \int_0^1 \mathrm{e}^{\alpha t} t^n \,\mathrm{d}t \\ &= \left[\mathrm{e}^{\alpha t} \frac{t^{n+1}}{n+1}\right]_{t=0}^{t=1} - \int_0^1 \alpha \mathrm{e}^{\alpha t} \frac{t^{n+1}}{n+1} \,\mathrm{d}t \\ &= \frac{\mathrm{e}^{\alpha}}{n+1} - \frac{\alpha}{n+1} I_{n+1} \\ &= \frac{\mathrm{e}^{\alpha} - \alpha I_{n+1}}{n+1}. \end{aligned}$$

3. Let $\ell = \lim\limits_{n\to+\infty} I_n$. By Question 1 we know that ℓ exists in $\mathbb{R}$. Then $\mathrm{e}^{\alpha} - \alpha I_{n+1} \underset{n\to+\infty}{\longrightarrow} \mathrm{e}^{\alpha} - \alpha\ell$ hence, using the relation obtained in Question 2:

$$\lim_{n\to+\infty} I_n = 0.$$

Since $\mathrm{e}^{\alpha} - \alpha I_n \underset{n\to+\infty}{\longrightarrow} \mathrm{e}^{\alpha} - \alpha\ell = \mathrm{e}^{\alpha} \neq 0$ we conclude $\mathrm{e}^{\alpha} - \alpha I_n \underset{n\to+\infty}{\sim} \mathrm{e}^{\alpha}$, hence

$$I_n \underset{n\to+\infty}{\sim} \frac{\mathrm{e}^{\alpha}}{n}.$$

Exercise 3. With the given substitution:

- $\mathrm{d}u = \dfrac{\mathrm{d}t}{2\sqrt{t}}$, hence $\mathrm{d}t = 2\sqrt{t}\,\mathrm{d}u = 2(u-1)\,\mathrm{d}u$;
- when $t = 1$, $u = 2$;
- when $t = a^2$, $u = 1 + \sqrt{a^2} = 1 + a$ since $a > 0$.

Then:

$$I = \int_2^{1+a} \frac{2(u-1)\,\mathrm{d}u}{u} = \int_2^{1+a} \left(2 - \frac{1}{u}\right) \mathrm{d}u = 2(1+a-2) - 2\big(\ln(1+a) - \ln 2\big) = 2(a-1) - 2\ln\left(\frac{1+a}{2}\right).$$

Exercise 4.

1. $\mathscr{B} = \left(1, X, X^2\right)$, $\dim E = 3$.

2. There are several ways to determine that $\mathscr{C}$ is a basis of E; later we need to determine some coordinates in $\mathscr{C}$, so we are going to show that the system associated with the coordinates in $\mathscr{C}$ possesses a unique solution: let $P = a + bX + cX^2 \in E$ and let $x, y, z \in \mathbb{R}$. Then:

$$\begin{aligned} P = xP_0 + yP_1 + zP_2 \quad &\iff \begin{cases} -x = b \\ x + y + z = c \\ -y + z = a \end{cases} \\ &\underset{R2 \leftarrow R_2 + R1}{\iff} \begin{cases} -x = b \\ y + z = b + c \\ -y + z = a \end{cases} \\ &\underset{R3 \leftarrow R_3 + R2}{\iff} \begin{cases} -x = b \\ y + z = b + c \\ +2z = a + b + c \end{cases} \\ &\iff \begin{cases} x = -b \\ y = b + c - z = -\dfrac{1}{2}a + \dfrac{1}{2}b + \dfrac{1}{2}c \\ z = \dfrac{1}{2}a + \dfrac{1}{2}b + \dfrac{1}{2}c \end{cases} \end{aligned}$$

We obtain a unique solution, hence $\mathscr{C}$ is a basis of E.

3. $$P = P_0 - P_1 + P_2 = X^2 - X + 2,$$

   and

   $$[P]_{\mathscr{B}} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

4. From Question 2:

   $$[P]_{\mathscr{C}} = \begin{pmatrix} -1 \\ 1/2 \\ 3/2 \end{pmatrix}.$$

Exercise 5.

1. Let $(x, y, z, t) \in F$. Then:

$$\begin{aligned}
(x,y,z,t) \in F &\iff \begin{cases} x+y+z+t=0 \\ 2x+y-z-t=0 \end{cases} \underset{R_2 \leftarrow R_2 - 2R_1}{\iff} \begin{cases} x+y+\ \ z+\ \ t=0 \\ \ \ -y-3z-3t=0 \end{cases} \\
&\iff \begin{cases} x = -y - z - t = 2z + 2t \\ y = -3z - 3t \\ z = z \\ t = t \end{cases} \\
&\iff (x, y, z, t) = z(2, -3, 1, 0) + t(2, -3, 0, 1).
\end{aligned}$$

   Hence a basis of F is $\mathscr{B} = \big((2, -3, 1, 0) + (2, -3, 0, 1)\big)$. We conclude that $\dim F = 2$.

2.
   - We first check that F and G are independent by checking that $F \cap G = \{0_E\}$: let $w \in F \cap G$. Since $w \in G = \operatorname{Span}\{u, v\}$, there exists $\alpha, \beta \in \mathbb{R}$ such that $w = \alpha u + \beta v$, i.e.,

     $$w = (\alpha - \beta, \alpha + \beta, \alpha, \alpha + \beta).$$

     Since $w \in F$, we must have:

     $$\begin{cases} (\alpha - \beta) + (\alpha + \beta) + \alpha + (\alpha + \beta) = 0 \\ 2(\alpha - \beta) + (\alpha + \beta) - \alpha - (\alpha - \beta) = 0 \end{cases}$$

     that is,

     $$\begin{cases} 4\alpha + \beta = 0 \\ \alpha = 0 \end{cases}$$

     hence $\alpha = \beta = 0$, hence $w = 0_E$. Hence $F \cap G = \{0_E\}$, hence F and G are independent.
   - We now show that $E = F + G$: since $G = \operatorname{Span}\{u, v\}$ and since u and v are not collinear, we conclude that (u, v) is a basis of G and hence $\dim G = 2$. Now, from Grassmann's Formula (and using the fact that F and G are independent):

     $$\dim(F \oplus G) = \dim F + \dim G = 2 + 2 = 4$$

     We conclude by the Inclusion–Equality Theorem: since $F \oplus G$ is a subspace of E and $\dim(F \oplus G) = 4 = \dim(E) < +\infty$, we must have $E = F \oplus G$.

Exercise 6. See lecture

Exercise 7.

1. a) Let $(x, y, z) \in \mathbb{R}^3$. Then:

$$(x, y, z) \in \operatorname{Ker} f \qquad \iff \qquad \begin{cases} x + y + z = 0 \\ 2x - y - z = 0 \\ -x + y + z = 0 \end{cases}$$

$$
\begin{aligned}
&\underset{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1}}{\iff} && \begin{cases} x + \ \ y + \ \ z = 0 \\ \quad - 3y - 3z = 0 \\ \qquad 2y + 2z = 0 \end{cases} \\
&\iff && \begin{cases} x + y + z = 0 \\ \qquad y + z = 0 \end{cases} \\
&\iff && \begin{cases} x = 0 \\ y = -z \\ z = z \end{cases} \\
&\iff && (x, y, z) = z(0, -1, 1).
\end{aligned}
$$

Hence $\operatorname{Ker} f = \operatorname{Span}\big\{(0, -1, 1)\big\}$, and a basis of $\operatorname{Ker} f$ is $\big((0, -1, 1)\big)$.

b) We know that a generating family of $\operatorname{Im} f$ is given by the image by f of a basis of $\mathbb{R}^3$. Hence:

$$\operatorname{Im} f = \operatorname{Span}\big\{f(1,0,0), f(0,1,0), f(0,0,1)\big\} = \operatorname{Span}\big\{(1,2,-1), (1,-1,1), (1,-1,1)\big\} = \operatorname{Span}\big\{(1,2,-1), (1,-1,1)\big\}$$

and since the two vectors that appear are not collinear, we conclude that a basis of $\operatorname{Im} f$ is:

$$\big((1,2,-1), (1,-1,1)\big).$$

2. f is not injective, not surjective, not bijective.

3. For a linear map $f : E \to F$, $\dim E = \operatorname{rk} f + \dim \operatorname{Ker} f$.

4. Here, $\dim \mathbb{R}^3 = 3$, $\operatorname{rk} f = \dim \operatorname{Im} f = 2$ and $\dim \operatorname{Ker} f = 1$, hence $\dim \mathbb{R}^3 \operatorname{rk} f + \dim \operatorname{Ker} f$.

5. a)

$$[f]_{\mathscr{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

b)

$$[f]_{\mathscr{C},\mathscr{B}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$