SCAN 1 — Math Test #5 — 2h

May 21, 2021

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. We define

$$\begin{array}{rccc} \varphi : & \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \displaystyle\int_x^{x+x^2} \frac{\cos(t)}{t^2}\,\mathrm{d}t. \end{array}$$

1. Explain why φ is well defined.
2. Show that φ is differentiable, and determine an expression of φ'.
3. Let $x \in \mathbb{R}_+^*$. Show that there exists $c_x \in \left[x, x+x^2\right]$ such that

$$\varphi(x) = \frac{\cos(c_x)}{1+x}.$$

4. Deduce the value of the limits $\lim\limits_{x \to 0^+} \varphi(x)$ and $\lim\limits_{x \to +\infty} \varphi(x)$.

Exercise 2. Throughout this exercise, α is a given constant real number. For $n \in \mathbb{N}$ we define:

$$I_n = \int_0^1 \mathrm{e}^{\alpha t} t^n \,\mathrm{d}t.$$

You're given that I_n is well defined.

1. Determine the variations of the sequence $(I_n)_{n \geq 0}$, and deduce that the sequence $(I_n)_{n \geq 0}$ converges.
2. For $n \in \mathbb{N}$, find a relation between I_n and I_{n+1}.
3. Deduce the limit ℓ of the sequence $(I_n)_{n \geq 0}$ as well as a simple equivalent of I_n as $n \longrightarrow +\infty$.

Exercise 3. Let $a \in \mathbb{R}_+^*$. Use the substitution $u = 1 + \sqrt{t}$ to compute the value of the following integral:

$$I = \int_1^{a^2} \frac{\mathrm{d}t}{1+\sqrt{t}}.$$

Exercise 4. Let $E = \mathbb{R}_2[X]$ be the real vector space of formal polynomials with real coefficients, indeterminate X, and of degree non-greater than 2.

1. Recall (without any justifications) the standard basis $\mathscr{B}$ of E as well as the dimension of E.
2. We define the following vectors of E:

$$P_0 = X^2 - X, \qquad P_1 = X^2 - 1, \qquad P_2 = X^2 + 1.$$

   Show that $\mathscr{C} = (P_0, P_1, P_2)$ is a basis of E.

3. Let $P \in E$ such that $[P]_{\mathscr{C}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Determine P and $[P]_{\mathscr{B}}$.
4. Let $P = 1 + X + X^2$. Determine $[P]_{\mathscr{C}}$.

Exercise 5. Let $E = \mathbb{R}^4$ and define:

$$F = \left\{(x, y, z, t) \in E \,\middle|\, x + y + z + t = 0,\ 2x + y - z - t = 0\right\}$$

and

$$u = (1, 1, 1, 1), \qquad v = (-1, 1, 0, 1)$$

and

$$G = \operatorname{Span}\{u, v\}.$$

You are given that F is a subspace of E.

1. Determine a basis of $\mathscr{B}$ of F, and the dimension of F.
2. Show that $E = F \oplus G$.

Exercise 6. Let E and F be two vector spaces over $\mathbb{K}$. Let $f : E \to F$ be a linear map. Show that $\operatorname{Ker} f$ is a subspace of E.

Exercise 7. Let

$$f : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x, y, z) & \longmapsto & (x + y + z, 2x - y - z, -x + y + z). \end{array}$$

You're given that f is a linear map.

1. a) Determine $\operatorname{Ker} f$. If $\operatorname{Ker} f \neq \{0_E\}$, determine a basis of $\operatorname{Ker} f$.
   b) Determine $\operatorname{Im} f$. If $\operatorname{Im} f \neq \{0_E\}$, determine a basis of $\operatorname{Im} f$.
2. Is f injective? surjective? bijective?
3. Recall the Rank–Nullity Theorem.
4. Check that the Rank–Nullity Theorem is satisfied for f.
5. Let $\mathscr{B}$ be the standard basis of $\mathbb{R}^3$, and let

$$\mathscr{C} = \big((1, 0, 0), (1, 1, 0), (1, 1, 1)\big).$$

   You're given that $\mathscr{C}$ is a basis of $\mathbb{R}^3$.

   a) Give the matrix $[f]_{\mathscr{B}}$ of f in $\mathscr{B}$.
   b) Give the matrix $[f]_{\mathscr{C},\mathscr{B}}$ of f in the bases $\mathscr{C}$ and $\mathscr{B}$.