SCAN 1 — Solution of Math Test #6

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Exercise 1.

1. Let $(x,y,z)\in E$. Then:

$$(x,y,z)\in F \iff (x,y,z)=y(-1,1,0)+z(-1,0,1).$$

Hence a basis of F is

$$\big((-1,1,0),(-1,0,1)\big).$$

Hence $\dim F=2$.

2. The spaces F and G are independent: let $u\in F\cap G$. Since $u\in G$, there exists $\lambda\in\mathbb{R}$ such that $u=\lambda(1,-1,1)=(\lambda,-\lambda,\lambda)$, and since $u\in F$ we must have $\lambda-\lambda+\lambda=0$, i.e., $\lambda=0$. Hence $u=0_E$.

Moreover, $\dim G=1$, hence by Grassmann's formula (and since F and G are independent), $\dim(F+G)=\dim F+\dim G=2+1=3$. Finally, we conclude that $E=F\oplus G$ by the Inclusion–Equality Theorem.

3. By the previous questions, we know that

$$\mathscr{B}=\big((-1,1,0),(-1,0,1),(1,-1,1)\big)$$

is a basis of E. We now compute the coordinates of $u=(4,1,-3)$ in $\mathscr{B}$:

$$[u]_{\mathscr{B}}=\begin{pmatrix}x\\y\\z\end{pmatrix} \iff \begin{cases}-x-y+z=4\\ x-z=1\\ y+z=-3\end{cases} \underset{R_2\leftarrow R_2+R1}{\iff} \begin{cases}-x-y+z=4\\ -y=5\\ y+z=-3\end{cases} \iff \begin{cases}x=-4-y+z=3\\ z=-3-y=2\\ y=-5\end{cases}$$

So that

$$\begin{aligned}u_F&=3(-1,1,0)-5(-1,0,1)=(2,3,-5)\in F\\ u_G&=2(1,-1,1)=(2,-2,2)\in G,\end{aligned}$$

and $u=u_F+u_G$.

Exercise 2.

1. $A=\begin{pmatrix}1&1&1\\0&1&-1\end{pmatrix}$.

2. We compute the rank of $[\mathscr{C}_E]_{\mathscr{B}_E}$:

$$\operatorname{rk}\mathscr{C}_E=\operatorname{rk}\begin{pmatrix}1&0&1\\2&1&0\\1&1&1\end{pmatrix} \underset{C_3\leftarrow C_3-C_1}{=} \begin{pmatrix}1&0&0\\2&1&-2\\1&1&0\end{pmatrix}=3 \qquad \textit{last two columns are not collinear}$$

$\mathscr{C}_E$ is a family of 3 vectors of rank 3, hence $\mathscr{C}_E$ is independent; and since $\dim E=3$, $\mathscr{C}_E$ is also a generating family; hence $\mathscr{C}_E$ is a basis of E.

To show that $\mathscr{C}_F$ is a basis of F, we show that the system associated to the coordinates in $\mathscr{C}_F$ possesses a unique solution:

$$\begin{cases}a-b=x\\2a+b=y\end{cases} \underset{R_2\leftarrow R_2-2R_1}{\iff} \begin{cases}a-b=x\\3b=-2x+y\end{cases} \iff \begin{cases}a=x+b=x/3+y/3\\ b=-2x/3+y/3\end{cases}$$

(Another method is to say that $\mathscr{C}_F$ consists of two non-collinear vectors in a space of dimension 2, but we'll need the solutions of the system later).

3. a)

$$A' = \begin{pmatrix} 5/3 & 2/3 & 1/3 \\ 7/3 & -4/3 & -5/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & 2 & 1 \\ 7 & -4 & -5 \end{pmatrix},$$

since

$$f(1,2,1) = (4,1), \qquad \text{and} \qquad \big[(4,1)\big]_{\mathscr{C}_F} = \begin{pmatrix} 5/3 \\ 7/3 \end{pmatrix}$$

$$f(0,1,1) = (2,0), \qquad \text{and} \qquad \big[(2,0)\big]_{\mathscr{C}_F} = \begin{pmatrix} 2/3 \\ -4/3 \end{pmatrix}$$

$$f(1,0,1) = (2,-1), \qquad \text{and} \qquad \big[(2,-1)\big]_{\mathscr{C}_F} = \begin{pmatrix} 1/3 \\ -5/3 \end{pmatrix}.$$

b) The change of basis formula states:

$$[f]_{\mathscr{C}_E,\mathscr{C}_F} = [\mathscr{C}_F]_{\mathscr{B}_F}^{-1}\,[f]_{\mathscr{B}_E,\mathscr{B}_F}\,[\mathscr{C}_E]_{\mathscr{B}_E}.$$

Now

$$[\mathscr{C}_E]_{\mathscr{B}_E} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad [\mathscr{C}_F]_{\mathscr{B}_F} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

To compute $[\mathscr{C}_E]_{\mathscr{B}_E}^{-1}$ we need to solve the following system (which is the one we solved earlier):

$$\begin{cases} a - b = x \\ 2a + b = y \end{cases} \iff \begin{cases} a = x + b = x/3 + y/3 \\ b = -2x/3 + y/3 \end{cases}$$

hence

$$[\mathscr{C}_F]_{\mathscr{B}_F}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}.$$

Finally,

$$A' = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & 2 & 1 \\ -7 & -4 & -5 \end{pmatrix}$$

Exercise 3. The characteristic polynomial of A is:

$$\begin{aligned}
\chi_A(\lambda) &= \det\begin{pmatrix} 6-\lambda & -4 & -3 \\ 3 & -1-\lambda & -3 \\ 4 & -4 & -1-\lambda \end{pmatrix} \\
&\underset{C_1 \leftarrow C_1 + C_2}{=} \det\begin{pmatrix} 2-\lambda & -4 & -3 \\ 2-\lambda & -1-\lambda & -3 \\ 0 & -4 & -1-\lambda \end{pmatrix} \\
&\underset{R_2 \leftarrow R_2 - R_1}{=} \det\begin{pmatrix} 2-\lambda & -4 & -3 \\ 0 & 3-\lambda & 0 \\ 0 & -4 & -1-\lambda \end{pmatrix} \\
&= (2-\lambda)\det\begin{pmatrix} 3-\lambda & 0 \\ -4 & -1-\lambda \end{pmatrix} \\
&= (2-\lambda)(3-\lambda)(-1-\lambda).
\end{aligned}$$

Hence the eigenvalues of A are -1, 2 and 3, all of multiplicity 1. We now determine the eigenspaces and eigenvectors:

- E_{-1}:

$$\begin{cases} 7x - 4y - 3z = 0 \\ 3x \quad\quad - 3z = 0 \\ 4x - 4y \quad\quad = 0 \end{cases} \underset{\substack{R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2/3}}{\iff} \begin{cases} 3x \quad\quad - 3z = 0 & \text{(useless)} \\ x \quad\quad - z = 0 \\ 4x - 4y \quad\quad = 0 \end{cases}$$

$$\iff \quad \begin{cases} z = x \\ y = x \\ x = x \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence we choose

$$X_{-1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

- E_2:

$$\begin{cases} 4x-4y-3z=0 \\ 3x-3y-3z=0 \\ 4x-4y-3z=0 \quad \text{(useless)} \end{cases} \underset{R_1 \leftrightarrow R_2}{\iff} \begin{cases} 3x-3y-3z=0 \\ 4x-4y-3z=0 \end{cases} \underset{R_2 \leftarrow R_2/3}{\iff} \begin{cases} x-\ y-\ z=0 \\ 4x-4y-3z=0 \end{cases}$$

$$\iff R_2 \leftarrow R_2 - 4R_1 \begin{cases} x-y-z=0 \\ z=0 \end{cases}$$

$$\iff \quad \begin{cases} x = y \\ z = 0 \\ y = y \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Hence we choose

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

- E_3:

$$\begin{cases} 3x-4y-3z=0 \\ 3x-4y-3z=0 \\ 4x-4y-4z=0 \end{cases} \iff \begin{cases} 3x-4y-3z=0 \\ 4x-4y-4z=0 \end{cases} \underset{R_2 \leftarrow R_2 - R_1}{\iff} \begin{cases} 3x-4y-3z=0 \\ x \quad -\ z=0 \end{cases}$$

$$\iff \quad \begin{cases} x = z \\ y = 0 \\ z = z \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence we choose

$$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We construct P and D as:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

and we have $D = P^{-1}AP$.

Exercise 4.

1.

$$\begin{aligned} \chi_A(\lambda) &= \det \begin{pmatrix} 4-\lambda & -2 & -5 \\ 0 & 2-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{pmatrix} \\ &\underset{C_1 \leftarrow C_1 + C_2}{=} \det \begin{pmatrix} 2-\lambda & -2 & -5 \\ 2-\lambda & 2-\lambda & 1 \\ 0 & -1 & -1-\lambda \end{pmatrix} \\ &\underset{R_2 \leftarrow R_2 - R_1}{=} \det \begin{pmatrix} 2-\lambda & -2 & -5 \\ 0 & 4-\lambda & 6 \\ 0 & -1 & -1-\lambda \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= (2-\lambda)\det\begin{pmatrix} 4-\lambda & 6 \\ -1 & -1-\lambda \end{pmatrix} \\
&= (2-\lambda)\big((4-\lambda)(-1-\lambda)+6\big) \\
&= (2-\lambda)\big(\lambda^2-3\lambda+2\big) = (2-\lambda)(\lambda-2)(\lambda-1) \\
&= -(\lambda-2)^2(\lambda-1).
\end{aligned}
$$

Hence 1 is an eigenvalue of A of multiplicity 1 and 2 is an eigenvalue of A of multiplicity 2.

2. The rank of $A-2I_3$ is:

$$\operatorname{rk}(A-2I_3) = \operatorname{rk}\begin{pmatrix} 2 & -2 & -5 \\ 0 & 0 & 1 \\ 1 & -1 & -3 \end{pmatrix} = 2$$

hence, by the Rank–Nullity Theorem, $\dim E_2 = 3-2 = 1 \neq$ multiplicity of 2. Hence A is not diagonalizable.

3.
   - For u:

$$A\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

   hence $f(u)=u$; since $u \neq 0_E$, we conclude that u is an eigenvector of f associated to 1.

   - For v:

$$A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

   hence $f(v)=2v$; since $v \neq 0_E$, we conclude that v is an eigenvector of f associated to 2.

4. $P = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

   We compute P^{-1} by solving the following linear system:

$$
\begin{cases} x+y+3z=a \\ -x+y=b \\ x+z=c \end{cases}
\underset{\substack{R_2 \leftarrow R_2+R_1 \\ R_3 \leftarrow R_3-R_1}}{\iff}
\begin{cases} x+y+3z=a \\ 2y+3z=a+b \\ -y-2z=-a+c \end{cases}
$$

$$
\underset{R_2 \leftarrow R_2+2R_3}{\iff}
\begin{cases} x+y+3z=a \\ -z=-a+b+2c \\ -y-2z=-a+c \end{cases}
$$

$$
\iff
\begin{cases} x = a-y-3z = -a+b+3c \\ y = -2z+a-c = -a+2b+3c \\ z = a-b-2c \end{cases}
$$

   Hence

$$P^{-1} = \begin{pmatrix} -1 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & -1 & -2 \end{pmatrix}.$$

5. We already know that $f(u)=u$, $f(v)=2v$, hence the first two columns of T are correct. Moreover,

$$A\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$$

   hence $f(w)=(7,1,2)=2w+v$. Hence the last column of T is also correct.

6. a) $N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ hence, for all $k \geq 2$, $N^k = \mathbf{0}$.

b) To apply the Binomial Theorem to $D+N$ we need to check that D and N commute:

$$DN = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad ND = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $DN = ND$, and we can apply the Binomial Theorem: for $n \geq 2$,

$$\begin{aligned} T^n = (D+N)^n &= \sum_{k=0}^{n} \binom{n}{k} D^{n-k} N^k \\ &= \binom{n}{0} D^n + \binom{n}{1} D^{n-1} N \qquad \textit{since for } k \geq 2,\ N^k = \mathbf{0} \\ &= D^n + nD^{n-1}N. \end{aligned}$$

Now,

$$D^{n-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{n-1} & 0 \\ 0 & 0 & 2^{n-1} \end{pmatrix} \qquad \text{hence } D^{n-1}N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2^{n-1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we conclude that

$$T^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}.$$

7. By the Change of Basis Formula, we know that $A = PTP^{-1}$, hence $A^n = PT^nP^{-1}$. Notice that we only need to compute the first column of A^n:

$$\begin{aligned} A^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= PT^n \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \\ &= P \begin{pmatrix} -1 \\ -2^n + n2^{n-1} \\ 2^n \end{pmatrix} \\ &= \begin{pmatrix} -1 - 2^n + n2^{n-1} + 3 \cdot 2^n \\ 1 - 2^n + n2^{n-1} \\ -1 + 2^n \end{pmatrix} \\ &= \begin{pmatrix} -1 + 2^{n+1} + n2^{n-1} \\ 1 - 2^n + n2^{n-1} \\ -1 + 2^n \end{pmatrix}. \end{aligned}$$

Hence

$$f^n(1,0,0) = \big(-1 + 2^{n+1} + n2^{n-1}, 1 - 2^n + n2^{n-1}, -1 + 2^n\big).$$

Exercise 5.

- If $A = \lambda_0 I_n$, then A is already diagonal, hence A is diagonalizable.
- If A is diagonalizable, there exists an invertible matrix $P \in M_n(\mathbb{R})$ such that $A = PDP^{-1}$ where $D = \lambda_0 I_n$. Hence $A = \lambda_0 P I_n P^{-1} = \lambda_0 PP^{-1} = \lambda_0 I_n$.

Exercise 6. Let $\lambda_0 \in \mathbb{K}$ be an eigenvalue of p. This means that

$$E_{\lambda_0} = \operatorname{Ker}(p - \lambda_0 \operatorname{id}_E) \neq \{0_E\}.$$

Let $x \in E_{\lambda_0}$ with $x \neq 0_E$. Then:

$$p(x) = \lambda_0 x$$

and

$$p^2(x) = p(\lambda_0 x) = \lambda_0 p(x) = \lambda_0^2 x.$$

Since $p^2 = p$ we must have

$$\lambda_0^2 x = \lambda_0 x.$$

Since $x \neq 0_E$, we must have $\lambda_0^2 = \lambda_0$, hence $\lambda_0 \in \{0, 1\}$.