SCAN 1 — Math Test #6 — 2h

June 15, 2021

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $E = \mathbb{R}^3$ and let

$$F = \{(x, y, z) \in E \mid x + y + z = 0\}$$

and let

$$G = \operatorname{Span}\{(1, -1, 1)\}.$$

You're given that F and G are subspaces of E.

1. Determine a basis of F, and deduce the dimension of F.
2. Show that $E = F \oplus G$.
3. From the previous question, we know that every vector $u \in E$ can be uniquely written as $u = u_F + u_G$ with $u_F \in F$ and $u_G \in G$. Find u_F and u_G for $u = (4, 1, -3)$.

Exercise 2. Let $E = \mathbb{R}^3$ and $F = \mathbb{R}^2$. Let $\mathscr{B}_E$ be the standard basis of E and let $\mathscr{B}_F$ be the standard basis of F. Let

$$f : \begin{array}{ccc} E & \longrightarrow & F \\ (x, y, z) & \longmapsto & (x + y + z, y - z). \end{array}$$

You're given that f is a linear map.

1. Determine the matrix $A = [f]_{\mathscr{B}_E, \mathscr{B}_F}$ of f in the bases $\mathscr{B}_E$, $\mathscr{B}_F$.
2. Let

$$\mathscr{C}_E = \big((1, 2, 1), (0, 1, 1), (1, 0, 1)\big)$$

and

$$\mathscr{C}_F = \big((1, 2), (-1, 1)\big).$$

Check that $\mathscr{C}_E$ is a basis of E and that $\mathscr{C}_F$ is a basis of F.

Note that, in the sequel, we won't need to determine coordinates in $\mathscr{C}_E$, but we'll need to determine some coordinates in $\mathscr{C}_F$; so you can choose the most appropriate method to show that $\mathscr{C}_E$ and $\mathscr{C}_F$ are bases.

3. Express the matrix $A' = [f]_{\mathscr{C}_E, \mathscr{C}_F}$ of f in the basis $\mathscr{C}_E$ and $\mathscr{C}_F$ in two different ways:
   a) Using a direct method (i.e., the definition of the matrix of a linear map in bases).
   b) Using the Change of Basis Formula (after stating the general formula).

Exercise 3. Let

$$A = \begin{pmatrix} 6 & -4 & -3 \\ 3 & -1 & -3 \\ 4 & -4 & -1 \end{pmatrix}.$$

Show that A is diagonalizable and find an invertible matrix $P \in M_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal. *Don't compute P^{-1}.*

Exercise 4. Let $E = \mathbb{R}^3$, let $\mathscr{B}$ be the standard basis of E, and let $f : E \to E$ be the endomorphism the matrix of which in the basis $\mathscr{B}$ is:

$$A = [f]_{\mathscr{B}} = \begin{pmatrix} 4 & -2 & -5 \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

1. Compute the characteristic polynomial of A to determine that 1 is an eigenvalue of A of multiplicity 1 and that 2 is an eigenvalue of A of multiplicity 2.
2. Explain why A is not diagonalizable.
3. Check that the vector $u = (1, -1, 1)$ is an eigenvector of f associated to 1, and that the vector $v = (1, 1, 0)$ is an eigenvector of A associated to 2.
4. We set $w = (3, 0, 1)$, and we define:

$$\mathscr{C} = (u, v, w).$$

Write matrix $P = [\mathscr{C}]_{\mathscr{B}}$ (i.e., the columns of P are the coordinates of the vectors of $\mathscr{C}$ in $\mathscr{B}$), show that P is invertible and compute P^{-1}. This shows that $\mathscr{C}$ is a basis of E.

5. Show that

$$[f]_{\mathscr{C}} = T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

6. Let D and N be the matrices

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so that $T = D + N$.

a) Compute N^2 and deduce, for $k \in \mathbb{N}$ with $k \geq 2$, the value of N^k.

b) Let $n \in \mathbb{N}$ with $n \geq 2$. Use the Binomial Theorem (after checking that you can apply it) to determine the value of T^n.

7. Determine, for $n \in \mathbb{N}$ with $n \geq 2$, the value of $f^n((1, 0, 0))$.

Exercise 5. Let $A \in M_n(\mathbb{R})$ be a matrix with a unique eigenvalue λ_0 of multiplicity n. Show that:

$$A \text{ is diagonalizable} \iff A = \lambda_0 I_n.$$

Exercise 6. Let E be a vector space over $\mathbb{K}$ and let $p : E \to E$ be an endomorphism such that $p^2 = p$ (i.e., $p \circ p = p$). What are the possible values for the eigenvalues of p?