INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #1

Romaric Pujol, romaric.pujol@insa-lyon.fr

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Exercise 1.

1. Let $n \in \mathbb{N}^*$. Then, for $k \in \{1, \dots, n\}$:

$$\begin{aligned} & n^2 + k^2 > n^2 \\ \text{hence} \quad & \sqrt{n^2+k^2} > n > 0 \\ \text{hence} \quad & \frac{1}{\sqrt{n^2+k^2}} < \frac{1}{n} \end{aligned}$$

and hence:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}} < \sum_{k=1}^{n} \frac{1}{n} = n\frac{1}{n} = 1.$$

2. Let $n \geq 2$ and $k \in \{2, \dots, n\}$. Then:

$$\begin{aligned} k(k-1)\binom{n}{k} &= k(k-1)\frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(k-2)!(n-k)!} \\ &= \frac{n!}{(k-2)!\big(n-2-(k-2)\big)!} \\ &= n(n-1)\frac{(n-2)!}{(k-2)!\big(n-2-(k-2)\big)!} \\ &= n(n-1)\binom{n-2}{k-2} \end{aligned}$$

Hence

$$\begin{aligned} \sum_{k=2}^{n} k(k-1)\binom{n}{k} &= \sum_{k=2}^{n} n(n-1)\binom{n-2}{k-2} \\ &= n(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k-2} \\ &= n(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k-2}1^{n-2-(k-2)}1^{k-2} \\ &= n(n-1)2^{n-2} \qquad \textit{by the Binomial Theorem} \end{aligned}$$

Exercise 2. Let $x \in \mathbb{R}$. Then:

$$\begin{aligned} 4\sin(x)\cos^2(x) - 3\sin(x) &= \sin(x)\big(4\cos^2(x) - 3\big) \\ &= 4\sin(x)\big(\cos^2(x) - 3/4\big) \\ &= 4\sin(x)\big(\cos^2(x) - 3/4\big) \end{aligned}$$

Hence:

$$\begin{aligned} 4\sin(x)\cos^2(x) - 3\sin(x) = 0 &\iff \sin(x) = 0 \text{ or } \cos(x) = \frac{\sqrt{3}}{2} \text{ or } \cos(x) = -\frac{\sqrt{3}}{2} \\ &\iff \sin(x) = 0 \text{ or } \cos(x) = \cos\left(\frac{\pi}{6}\right) \text{ or } \cos(x) = \cos\left(\frac{5\pi}{6}\right) \end{aligned}$$

$$\iff x \in \{0, \pi\} \text{ or } x \in \left\{-\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}\right\} \text{ or } x \in \left\{-\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}\right\}$$
$$\iff x \in \left\{-\frac{5\pi}{6}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}.$$

Exercise 3.

1. Let $x \in \mathbb{R}$. Then:

$$\begin{aligned}\text{The expression } -f(1-2x) \text{ is well defined} &\iff 1-2x \in [0,1]\\ &\iff 0 \le 1-2x \le 1\\ &\iff -1 \le -2x \le 0\\ &\iff 0 \le 2x \le 1\\ &\iff 0 \le x \le \frac{1}{2}.\end{aligned}$$

   Hence $D = [0, 1/2]$.

2. We can define the following intermediate functions:

$$\begin{aligned} f_1 : [0,1] &\longrightarrow \mathbb{R} & f_2 : [0,1] &\longrightarrow \mathbb{R}\\ x &\longmapsto f(-x), & x &\longmapsto f_1(x-1) = f(1-x),\\ f_3 : [0,1/2] &\longrightarrow \mathbb{R} & g : [0,1/2] &\longrightarrow \mathbb{R}\\ x &\longmapsto f_2(2x) = f(1-2x), & x &\longmapsto -f_3(x) = -f(1-2x).\end{aligned}$$

   The graph of g is obtained from that of f by performing the following operations (in this order):

   - flip the graph of f about the y-axis to obtain the graph of f_1,
   - shift the graph of f_1 to the right by 1 unit to obtain the graph of f_2,
   - squeeze the graph of f_2 horizontally by a factor of 2 to obtain the graph of f_3,
   - flip the graph of f_3 about the x-axis to obtain the graph of g.

3. See Figure 1.

Exercise 4.

1. a) Let $x \in \mathbb{R}^*$. Then:

$$p(x) = \frac{(2x+1)^2}{16x} - \frac{1}{4} = \frac{4x^2+4x+1}{16x} - \frac{4x}{16x} = \frac{4x^2+1}{16x}.$$

   We notice that p is the quotient of an even and an odd function, hence p is odd. More precisely: for $x \in \mathbb{R}^*$,

$$p(-x) = \frac{4(-x)^2+1)}{16(-x)} = -\frac{4x^2+1}{16x} = -p(x).$$

   b) i) Let $x, y \in [1/2, +\infty)$ such that $x < y$. Since $1/2 \le x < y$ we conclude that $xy > 1/4$ i.e., $4xy - 1 > 0$, and that $x - y < 0$, hence

$$f(x) - f(y) = \frac{(4xy-1)(x-y)}{16xy} < 0,$$

   hence $f(x) < f(y)$. This shows that f is increasing on $[1/2, +\infty)$.
   Let $x, y \in (0, 1/2]$ such that $x < y$. Since $0 < x < y \le 1/2$ we conclude that $xy < 1/4$ i.e., $4xy - 1 < 1$, and that $x - y < 0$, and that $xy > 0$, hence

$$f(x) - f(y) = \frac{(4xy-1)(x-y)}{16xy} > 0,$$

   hence $f(x) > f(y)$. This shows that f is decreasing on $(0, 1/2]$.

   ii) Since $p = f + 1/4$, the functions p and f have the same variations. We know that f (hence p) is:
   - increasing on $[1/2, +\infty)$,
   - decreasing on $(0, 1/2]$.

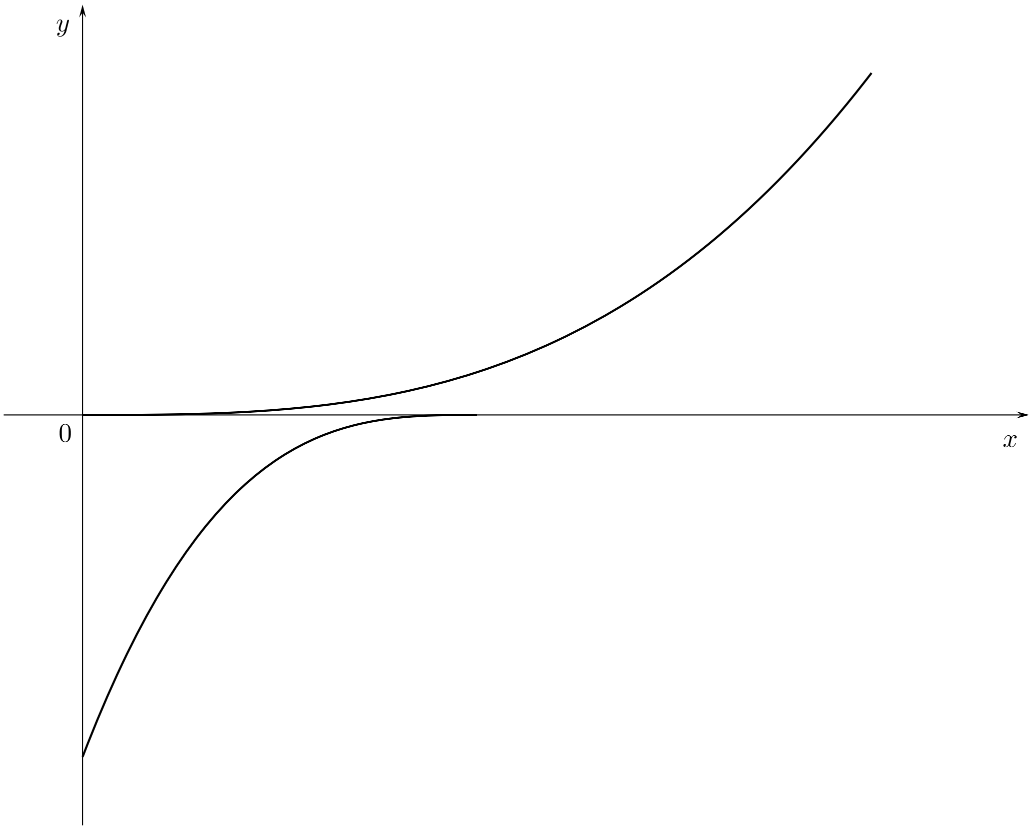

Figure 1 – Graph of f (above) and g (below) of Exercise 3.

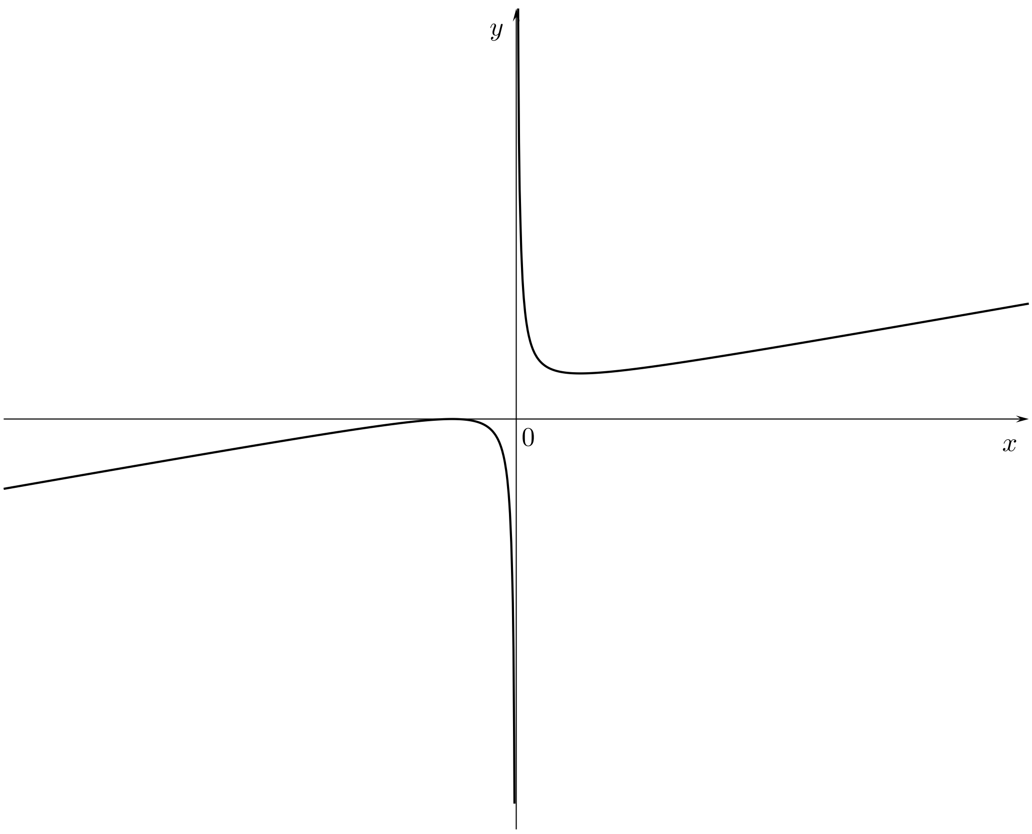

Figure 2 – Graph of function f of Exercise 4.

Since p is odd, we conclude that p (hence f) is:

- increasing on $(-\infty, -1/2]$,
- decreasing on $[-1/2, 0)$.

c) See Figure 2.

d)

$$A = [1/2, +\infty), \qquad B = (-\infty, 0] \cup (1/2, +\infty), \qquad C = \mathbb{R}_-,$$
$$D = \emptyset, \qquad E = \left\{-\frac{1}{2}\right\}, \qquad F = \left\{-\frac{1}{2}\right\} \cup \mathbb{R}_+^*.$$

2. a) We know that f is increasing on $[1/2, +\infty)$, hence

$$\forall x \in \left[\frac{1}{2}, +\infty\right), \ f(x) \geq f\left(\frac{1}{2}\right) = \frac{1}{2},$$

this show that g is well-defined. We also know that f is increasing on $[1/2, +\infty)$, hence g is increasing, hence g is injective.

b) Let $x \in [1/2, +\infty)$, $y \in [1/2, +\infty)$. Then:

$$\begin{aligned} g(x) = y &\iff \frac{(2x+1)^2}{16x} = y \\ &\iff (2x+1)^2 = 16xy \\ &\iff 4x^2 + 4x + 1 = 16xy \\ &\iff x^2 + x + \frac{1}{4} = 4xy \\ &\iff x^2 + (1-4y)x + \frac{1}{4} = 0 \end{aligned}$$

We recognize a quadratic, the discriminant of which is

$$\Delta = (1-4y)^2 - 1 = 16y^2 - 8y = 8y(2y-1) \geq 0,$$

and the solutions are:

$$\frac{4y - 1 \pm \sqrt{\Delta}}{2} = 2y - \frac{1}{2} \pm \sqrt{2y(2y-1)}.$$

We now show that there's exactly one solution $x \in [1/2, +\infty)$:

- if $y = 1/2$ then $\Delta = 0$, the quadratic possesses a unique solution which is $1/2$;
- if $y > 1/2$ then $2y - 1/2 > 1/2$, and so $2y - 1/2 + \sqrt{2y(2y-1)} \in (1/2, +\infty)$. Since g is injective, the other solution of the quadratic can't be in $[1/2, +\infty)$. We can check that explicitly too:

$$\begin{aligned} 2y - 1/2 - \sqrt{2y(2y-1)} &= \frac{(2y - 1/2 - \sqrt{2y(2y-1)})(2y - 1/2 + \sqrt{2y(2y-1)})}{2y - 1/2 + \sqrt{2y(2y-1)}} \\ &= \frac{(2y-1/2)^2 - 2y(2y-1)}{2y - 1/2 + \sqrt{2y(2y-1)}} \\ &= \frac{1/4}{2y - 1/2 + \sqrt{2y(2y-1)}} \end{aligned}$$

  and since the denominator is greater that $1/2$, we have

$$2y - 1/2 - \sqrt{2y(2y-1)} < 1/2.$$

Finally, we have proved that:

$$g(x) = y \iff x = 2y - 1/2 - \sqrt{2y(2y-1)} < 1/2.$$

We hence conclude that g is a bijection and that:

$$\begin{aligned} g^{-1} \ : \ [1/2, +\infty) &\longrightarrow [1/2, +\infty) \\ y &\longmapsto 2y - 1/2 - \sqrt{2y(2y-1)}. \end{aligned}$$

3. a) The sequence $(u_n)_{n\in\mathbb{N}}$ is well defined since
   - u_0 belongs to the domain of g (so that u_1 can be computed)
   - the domain and codomain of g are equal (hence we can compute u_{n+1} from u_n).

   b) Since g is increasing, the sequence $(u_n)_{n\in\mathbb{N}}$ is monotone, and its variations depend on its first terms. Now,

$$u_1 - u_0 = \frac{(2u_0+1)^2}{16u_0} - u_0 = \frac{4u_0^2+4u_0+1}{16u_0} - u_0 = \frac{-12u_0^2+4u_0+1}{16u_0} = -\frac{(6u_0+1)(2u_0-1)}{16u_0} < 0,$$

   hence $u_1 < u_0$, hence the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

Exercise 5. Since f is a polynomial with real coefficients, we know that if $1+2i$ is a root of f of multiplicity at least 2 if and only if $1-2i$ is a root of multiplicity at least 2, if and only if $f(x)$ can be factored by

$$g(x) = (x-1-2i)^2(x-1+2i)^2 = (x^2-2x+5)^2 = x^4-4x^3+14x^2-20x+25.$$

After a long division (only 2 steps) we find:

$$f(x) = (2x-1)g(x),$$

which shows that $1+2i$ and $1-2i$ are roots of f of multiplicity exactly 2.
The factored form of f in $\mathbb{C}$ and $\mathbb{R}$ are:

$$f(x) = 2\left(x-\frac{1}{2}\right)(x-1-2i)^2(x-1+2i)^2 \qquad \text{(in } \mathbb{C}\text{)}$$

$$f(x) = 2\left(x-\frac{1}{2}\right)(x^2-2x+5)^2 \qquad \text{(in } \mathbb{R}\text{)}$$

Finally, the roots of f and their multiplicity are:

- $1+2i$ of multiplicity 2;
- $1-2i$ of multiplicity 2;
- $1/2$ of multiplicity 1.