SCAN 1 — Solution of Math Test #2

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Exercise 1.

1. Let $x \in \mathbb{R}$. Then
$$A \text{ is well-defined} \iff x^2 - 1 > 0 \iff x^2 > 1 \iff x \in (-\infty, -1) \cup (1, +\infty).$$
Hence $D = (-\infty, -1) \cup (1, +\infty)$.

2. f is even: D is symmetric with respect to the origin, and for $x \in D$ one has:
$$f(-x) = \cosh\Big(\ln\big((-x)^2 - 1\big)\Big) = \cosh\Big(\ln\big((x^2 - 1\big)\Big) = f(x).$$

3. Since $\cosh \geq 1$, f is bounded from below and 1 is a lower bound of f.
Moreover, $f(\sqrt{2}) = \cosh\big(\ln(1)\big) = \cosh(0) = 1$, and we conclude that $\inf f = 1$.
Since $f(\sqrt{2}) = 1 = \inf f$, we conclude that $\min f$ exists and $\min f = 1$.
We now determine the values of $x \in D$ such that $f(x) = 1$: let $x \in D$. Then:
$$f(x) = 1 \iff \cosh\Big(\ln\big(x^2 - 1\big)\Big) \iff \ln\big(x^2 - 1\big) = 0 \iff x^2 - 1 = 1 \iff x^2 = 2 \iff x = \pm\sqrt{2}.$$

4. f is:
   - decreasing on $(1, \sqrt{2}]$: let $x, y \in (1, \sqrt{2}]$ such that $x < y$. Then:
$$\begin{aligned} 1 < x < y \leq \sqrt{2} &\implies 1 < x^2 < y^2 \leq 2 && \textit{since the square function is increasing on } \mathbb{R}_+ \\ &\implies 0 < x^2 - 1 < y^2 - 1 \leq 1 \\ &\implies \ln\big(x^2 - 1\big) < \ln\big(y^2 - 1\big) \leq 0 && \textit{since } \ln \textit{ is increasing} \\ &\implies \cosh\Big(\ln\big(x^2 - 1\big)\Big\rangle \cosh\Big(\ln\big(y^2 - 1\big)\Big) && \textit{since } \cosh \textit{ is decreasing on } \mathbb{R}_- \end{aligned}$$
   hence $f(x) > f(y)$, hence f is decreasing on $(1, \sqrt{2}]$.
   - increasing on $[\sqrt{2}, +\infty)$: let $x, y \in [\sqrt{2}, +\infty)$ such that $x < y$. Then:
$$\begin{aligned} \sqrt{2} \leq x < y &\implies 2 \leq x^2 < y^2 && \textit{since the square function is increasing on } \mathbb{R}_+ \\ &\implies 1 \leq x^2 - 1 < y^2 - 1 \\ &\implies 0 \leq \ln\big(x^2 - 1\big) < \ln\big(y^2 - 1\big) \\ &\implies \cosh\Big(\ln\big(x^2 - 1\big)\Big) < \cosh\Big(\ln\big(y^2 - 1\big)\Big) && \textit{since } \cosh \textit{ is increasing on } \mathbb{R}_+ \end{aligned}$$
   hence $f(x) < f(y)$, hence f is increasing on $[\sqrt{2}, +\infty)$,

   and by the parity of f:
   - increasing on $[-\sqrt{2}, -1)$,
   - decreasing on $(-\infty, -\sqrt{2}]$.

5. - $\lim\limits_{x \to +\infty} f(x)$: we know that:
$$x^2 - 1 \underset{x \to +\infty}{\longrightarrow} +\infty \qquad \text{and} \qquad \ln(X) \underset{X \to +\infty}{\longrightarrow} +\infty$$
   hence, by composition of limits,
$$\ln\big(x^2 - 1\big) \underset{x \to +\infty}{\longrightarrow} +\infty$$
   and since $\lim_{+\infty} \cosh = +\infty$ we obtain, by composition of limits,
$$\lim_{x \to +\infty} f(x) = +\infty.$$

- $\lim_{x\to 1^+} f(x)$: we know that

$$x^2-1 \underset{x\to 1^+}{\longrightarrow} 0 \qquad \text{and} \qquad \ln(X) \underset{X\to 0^+}{\longrightarrow} -\infty$$

hence, by composition of limits,

$$\ln\big(x^2-1\big) \underset{x\to 1^+}{\longrightarrow} -\infty$$

and since $\lim_{-\infty} \cosh = +\infty$ we obtain, by composition of limits,

$$\lim_{x\to 1^+} f(x) = +\infty.$$

- By parity of f we deduce:

$$\lim_{x\to -\infty} f(x) = +\infty \qquad \text{and} \qquad \lim_{x\to -1^-} f(x) = +\infty.$$

6. Since $\lim_{+\infty} f = +\infty$ we conclude that f is not bounded from above, and that

$$\sup f = +\infty.$$

7. Let $x \in D$. Then:

$$\begin{aligned} f(x) &= \cosh\Big(\ln\big(x^2-1\big)\Big) \\ &= \frac{1}{2}\left(\exp\Big(\ln\big(x^2-1\big)\Big) + \exp\Big(-\ln\big(x^2-1\big)\Big)\right) \\ &= \frac{1}{2}\left(x^2-1+\frac{1}{x^2-1}\right) \\ &= -\frac{1}{2}+\frac{1}{2}x^2+\frac{1}{2}\frac{1}{x^2-1}. \end{aligned}$$

Hence $\alpha = -1/2$, $\beta = 1/2$ and $\gamma = 1/2$.

8. See Figure 3.

Exercise 2.

1. B is well-defined since the domain of arcsinh is $\mathbb{R}$. For A: since $x^2 \geq 0$ we conclude that $2x^2+1 \geq 1$, and since the domain of cosh is $[1,+\infty)$, we conclude that A is well-defined too.

2. We have $x = \sinh(B)$ and hence, by the addition formula:

$$2x^2+1 = 2\sinh^2(B)+1 = \cosh(2B).$$

Hence

$$A = \operatorname{arccosh}\big(\cosh(2B)\big) = 2|B|.$$

3.
$$C = \sinh\left(\frac{1}{2}A\right) = \sinh\big(|B|\big) = \sinh\big|\operatorname{arcsinh}(x)\big|$$

Since sinh is odd and $\sinh(\mathbb{R}_+) \subset \mathbb{R}_+$, it is easy to see that:

$$\forall X \in \mathbb{R},\ \sinh\big(|X|\big) = \big|\sinh(X)\big|,$$

so that

$$C = \sinh\left(\frac{1}{2}A\right) = \sinh\big(|B|\big) = \Big|\sinh\big(\operatorname{arcsinh}(x)\big)\Big| = |x|.$$

Exercise 3.

1. First observe that the inequality about E can be rewritten as:

$$\forall t \in \mathbb{R},\ t-1 < E(t) \leq t.$$

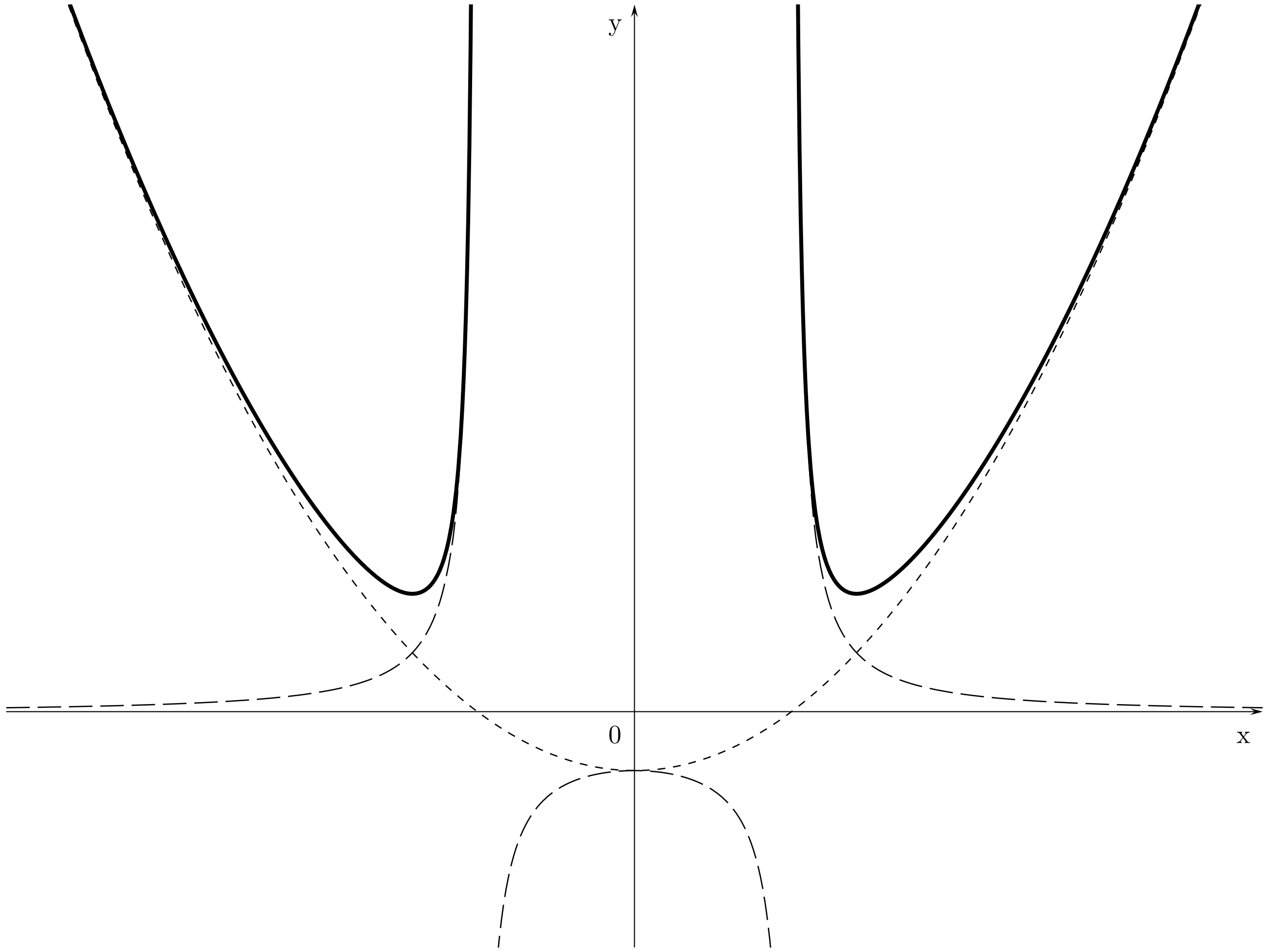

Figure 3 – Graph of f and the curves (C_1) (short dashes) and (C_2) (long dashes) of Exercise 1.

Let $n \in \mathbb{N}^*$. Then:

$$\sum_{k=1}^{n}(kx-1) < \sum_{k=1}^{n} E(kx) \leq \sum_{k=1}^{n} kx.$$

so that:

$$\left(x\sum_{k=1}^{n} k\right) - n < \sum_{k=1}^{n} E(kx) \leq x\sum_{k=1}^{n} k.$$

Now we know that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

hence

$$x\frac{n(n+1)}{2} - n < \sum_{k=1}^{n} E(kx) \leq x\frac{n(n+1)}{2}.$$

and hence:

$$x\frac{n(n+1)}{n^2} - \frac{2}{n} < \sum_{k=1}^{n} E(kx) \leq x\frac{n(n+1)}{n^2}.$$

So $\alpha_n = -2/n$.

2. For $n \in \mathbb{N}^*$,

$$\frac{n(n+1)}{n^2} = \frac{n^2+n}{n^2} = 1 + \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 1,$$

and

$$\alpha_n = -\frac{2}{n} \underset{n\to+\infty}{\longrightarrow} 0,$$

hence

$$\lim_{n\to+\infty} x\frac{n(n+1)}{n^2} - \frac{2}{n} = \lim_{n\to+\infty} x\frac{n(n+1)}{n}^2 = x,$$

and we conclude, by the Squeeze Theorem, that $\lim\limits_{+\infty} u_n$ exists and its value is x.

Exercise 4.

1. a)

$$\ell_1 = \lim_{x\to 0^+} x^\beta = \begin{cases} 0 & \text{if } \beta > 0 \\ 1 & \text{if } \beta = 0 \\ +\infty & \text{if } \beta < 0 \end{cases} \qquad \text{and} \qquad \ell_2 = \lim_{x\to+\infty} x^\beta = \begin{cases} 0 & \text{if } \beta < 0 \\ 1 & \text{if } \beta = 0 \\ +\infty & \text{if } \beta > 0. \end{cases}$$

b) Let $x \in (0, \pi/2)$ (which is a punctured right-sided neighborhood of 0). Then:

$$\frac{\ln(1+\sin x)}{x^\alpha} = \frac{\ln(1+\sin x)}{\sin x}\frac{\sin x}{x}x^{\alpha-1}$$

We know that:

$$\lim_{X\to 0}\frac{\ln(1+X)}{X} = 1,$$

and since $\sin x \underset{x\to 0^+}{\longrightarrow} 0$ we conclude, by the composition of limits theorem,

$$\lim_{x\to 0^+}\frac{\ln(1+\sin x)}{\sin x} = 1.$$

We also know that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1,$$

hence:

$$\lim_{x\to 0^+}\frac{\ln(1+\sin x)}{x^\alpha} = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ +\infty & \text{if } \alpha > 1. \end{cases}$$

2. Let $x \in \mathbb{R}$. Then:

$$f(x)^2 \geq 0$$

hence

$$f(x)^2 + 1 \geq 1$$

hence

$$0 < \frac{1}{f(x)^2+1} \leq 1,$$

which show that the function $x \mapsto \dfrac{1}{1+f(x)^2}$ is bounded.

- Since $\lim\limits_{x\to+\infty} x = +\infty$ we conclude, by (a corollary of) the Squeeze Theorem:

$$\ell_1 = \lim_{x\to+\infty}\frac{1}{1+f(x)^2} = +\infty.$$

- Since $\lim\limits_{x\to 0} x = 0$ we conclude, by (a corollary of) the Squeeze Theorem:

$$\ell_2 = \lim_{x\to 0}\frac{1}{1+f(x)^2} = 0.$$

Exercise 5. It is false: for example with $f = \tanh$ and $g = \tanh + 1$. Then, clearly, $f < g$, f is bounded from above and $\sup f = 1$, g is bounded from below and $\inf g = 0$ and the inequality

(P) $$\sup f = 1 \leq \inf g = 0$$

is false. See Figure 4.

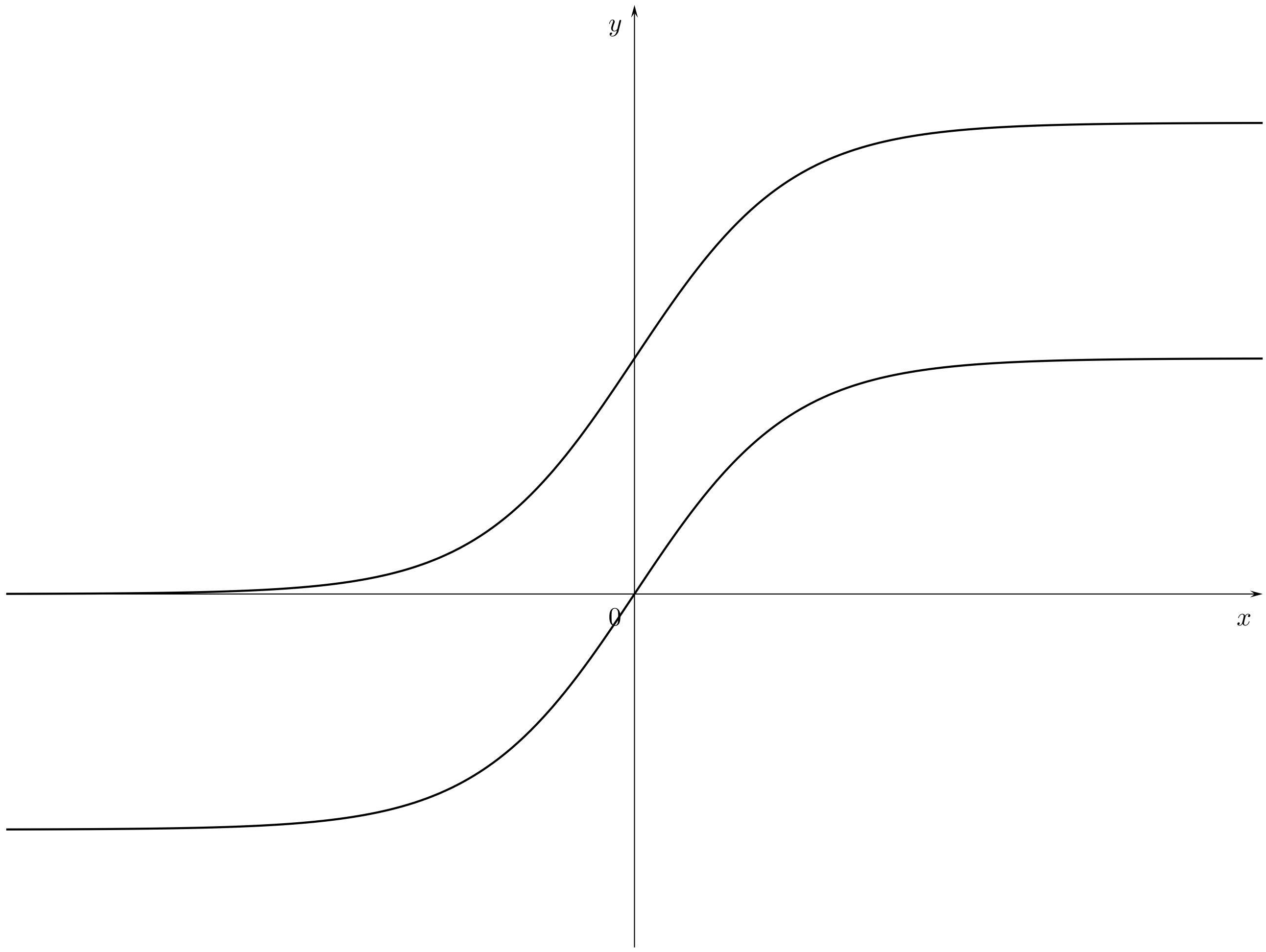

Figure 4 – Graph of f and g where $f < g$, f is bounded from above and g is bounded from below, and $\sup f < \inf g$.