INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #2 — 1h30

December 17, 2021

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1.

1. Find the maximum subset D of $\mathbb{R}$ such that for all $x \in D$, the following expression is defined:

$$A = \cosh\Big(\ln\big(x^2 - 1\big)\Big).$$

We hence define the function:

$$\begin{array}{rccl} f : & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \cosh\Big(\ln\big(x^2-1\big)\Big). \end{array}$$

2. Is f odd, even or neither? justify your answer.
3. Show that f is bounded from below, and determine $\inf f$. Does $\min f$ exist? if it does, what is its value? and for what values in D is it attained?
4. What are the variations of f? Justify your answer.

   Note. The previous question will help you guess what the variations of f are. You will get partial credit if you give a correct guess, and full credit if you can fully justify your answer.
5. Determine the limit of f at all the endpoints of D.
6. Is f bounded from above? what is the value of $\sup f$?
7. Let $x \in D$. Show that $f(x)$ can be written in the form:

$$f(x) = \alpha + \beta x^2 + \frac{\gamma}{x^2 - 1}$$

   where $\alpha, \beta, \gamma \in \mathbb{R}$ are coefficients you will determine.
8. Sketch, on the same figure, the graph of f and the curves

$$(C_1) \quad y = \alpha + \beta x^2 \qquad \text{and} \qquad (C_2) \quad y = \frac{\gamma}{x^2 - 1}.$$

Exercise 2. Let $x \in \mathbb{R}$ and set:

$$A = \operatorname{arccosh}(2x^2 + 1) \qquad \text{and} \qquad B = \operatorname{arcsinh}(x).$$

1. Briefly explain why A and B are well defined.
2. Find a relation between A and B.
3. Deduce a simpler expression for

$$C = \sinh\left(\frac{1}{2}\operatorname{arccosh}\big(2x^2 + 1\big)\right).$$

Exercise 3. In this exercise, we denote by E the integer part function. We recall that:

$$\text{for } t \in \mathbb{R},\ E(t) \text{ is the unique element of } \mathbb{Z} \text{ such that } E(t) \leq t < E(t) + 1.$$

From now on we fix a value $x \in \mathbb{R}$. With this value, we define the sequence $(u_n)_{n\in\mathbb{N}^*}$ as follows:

$$\forall n \in \mathbb{N}^*,\ u_n = \frac{2}{n^2}\sum_{k=1}^{n} E(kx).$$

1. Show that:

$$\forall n \in \mathbb{N}^*,\ x\frac{n(n+1)}{n^2} + \alpha_n < u_n \leq x\frac{n(n+1)}{n^2}$$

   where $\alpha_n \in \mathbb{R}$ that you will determine.

   Hint: the inequalities defining E will help you determine inequalities about u_n that involve the sum of an arithmetic progression.

2. Deduce that the limit $\ell = \lim\limits_{n\to+\infty} u_n$ exists in $\mathbb{R}$, and determine its value.

Exercise 4. The questions of this exercise are independent from each other.

1. a) Let $\beta \in \mathbb{R}$. Recall (without any justifications) the value of the following limits:

$$\ell_1 = \lim_{x\to 0^+} x^\beta \qquad \text{and} \qquad \ell_2 = \lim_{x\to+\infty} x^\beta.$$

   b) Let $\alpha \in \mathbb{R}$. Compute the value of the following limit:

$$\ell = \lim_{x\to 0^+} \frac{\ln(1+\sin x)}{x^\alpha}$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Compute the value of the following limits:

$$\ell_1 = \lim_{x\to+\infty} x + \frac{1}{1+f(x)^2} \qquad \text{and} \qquad \ell_2 = \lim_{x\to 0} \frac{x}{1+f(x)^2}$$

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R},\ f(x) < g(x).$$

We assume that g is bounded from below, and that f is bounded from above.

Is the following proposition true or false?

(P) $$\sup f \leq \inf g$$

If it is true, provide a full proof; if it is false, provide a full counterexample (i.e., specify a formula for f and g, and sketch the graph of f and g on the same figure).