SCAN 1 — Solution of Math Test #3

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Exercise 1.

1. For $t \in \mathbb{R}_+^*$,

$$f(t) = \exp\bigl(\sin(t)\ln(t)\bigr)$$

hence, by the Chain Rule and the Product Rule f is differentiable and:

$$\forall t \in \mathbb{R}_+^*,\ f'(t) = \left(\cos(t)\ln(t) + \frac{\sin(t)}{t}\right) t^{\sin(t)}.$$

2. Since $\sin(x) \underset{x\to 0}{\longrightarrow} 0$,

$$\mathrm{e}^{\sin(x)} - 1 \underset{x\to 0}{\sim} \sin(x) \underset{x\to 0}{\sim} x$$

and hence:

$$\frac{\mathrm{e}^{\sin(x)} - 1}{\cos(x) - 1}\arctan(x) \underset{x\to 0}{\sim} \frac{x}{-x^2/2}x = -\frac{1}{2} \underset{x\to 0}{\longrightarrow} -\frac{1}{2}.$$

Hence $\ell = -1/2$.

3.
   - If $\alpha > 0$ then $x^\alpha \underset{x\to+\infty}{\longrightarrow} +\infty$ hence $1 + x^\alpha \underset{x\to+\infty}{\sim} x^\alpha$ hence $x^{-\alpha}\bigl(1 + x^\alpha\bigr) \underset{x\to+\infty}{\sim} 1$.
   - If $\alpha = 0$, then $x^{-\alpha}\bigl(1 + x^\alpha\bigr) = 2 \underset{x\to+\infty}{\sim} 2$.
   - If $\alpha < 0$, then $x^\alpha \underset{x\to+\infty}{\longrightarrow} 0$, hence $x^{-\alpha}\bigl(1 + x^\alpha\bigr) \underset{x\to+\infty}{\sim} x^{-\alpha}$.

4. We know that

$$\ln(1+x) \underset{x\to 0}{=} x - \frac{x^2}{2} + \frac{x^3}{3} + o\bigl(x^3\bigr)$$
$$\mathrm{e}^x \underset{x\to 0}{=} 1 + x + \frac{x^2}{2} + o\bigl(x^2\bigr),$$

so that

$$\mathrm{e}^x\ln(1+x) \underset{x\to 0}{=} \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o\bigl(x^3\bigr)\right)\left(1 + x + \frac{x^2}{2} + o\bigl(x^2\bigr)\right)$$
$$\underset{x\to 0}{=} x + \frac{x^2}{2} + \frac{x^3}{3} + o\bigl(x^3\bigr)$$

hence

$$\frac{\mathrm{e}^x\ln(1+x) - x - x^2/2}{x^3} \underset{x\to 0}{=} \frac{1}{3} + o(1) \underset{x\to 0}{\longrightarrow} \frac{1}{3}.$$

Exercise 2.

1. If f is of class C^N on $[a,b]$ and $N+1$ times differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f(b) = \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!}(b-a)^k \ + \ \frac{f^{(N+1)}(c)}{(N+1)!}(b-a)^{N+1}.$$

2. In the case $f = \cos$, $a = 0$, $b = x \in (0, \pi/2]$ and $N = 5$ (which is valid since cos is of class C^∞ on $\mathbb{R}$, we conclude that there exists $c \in (0,x)$ such that

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{\cos(c)}{6!}x^6.$$

For P we take:

$$P(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

and we indeed have $\deg P = 4 \leq 5$.

Now from

$$0 < c < x \leq \frac{\pi}{2},$$

applying cos which is decreasing on $[0, \pi/2]$ yields

$$1 > \cos(c) > \cos(x) \geq 0,$$

and hence, since $-x^6/6! < 0$:

$$-\frac{x^6}{6!} < -\frac{\cos(c)}{6!}x^6 < 0$$

and finally:

$$P(x) - \frac{x^6}{6!} < P(x) - \frac{\cos(c)}{6!}x^6 = \cos(x) < P(x)$$

as required with $\alpha = 1/6!$.

3. From the given values:

$$0.944958 < P\left(\frac{1}{3}\right) < 0.944959$$

and hence

$$0.944956 < P\left(\frac{1}{3}\right) - \frac{1}{6!}\left(\frac{1}{3}\right)^6.$$

Applying the previous result with $x = 1/3$ and these values yields

$$0.944956 < \cos\left(\frac{1}{3}\right) < 0.944959$$

from which we deduce:

$$\cos\left(\frac{1}{3}\right) = 0.94495\ldots$$

(and the next digit is either a 6, a 7 or an 8).

Exercise 3.

1. Let $a, b \in \mathbb{R}$ with $a \neq b$, let $f : [a, b] \to \mathbb{R}$ such that f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

2. We proceed by contradiction: we assume that there exists $a, b \in \mathbb{R}$ such that $a \neq b$ and $f(a) = a$ and $f(b) = b$. Since f is differentiable on $\mathbb{R}$, f is continuous on $[a, b]$ and differentiable on (a, b) hence, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1,$$

which is impossible.

Exercise 4.

1.
   - On $[x_1, x_2]$: since f is twice differentiable on $[x_1, x_3]$, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), and since $f(x_1) = f(x_2)$ we conclude, by Rolle's Theorem, that there exists $c_1 \in (x_1, x_2)$ such that $f'(c_1) = 0$.
   - On $[x_2, x_3]$: since f is twice differentiable on $[x_1, x_3]$, f is continuous on $[x_2, x_3]$ and differentiable on (x_2, x_3), and since $f(x_2) = f(x_3)$ we conclude, by Rolle's Theorem, that there exists $c_2 \in (x_2, x_3)$ such that $f'(c_2) = 0$.
   - On $[c_1, c_2]$: since f is twice differentiable on $[x_1, x_3]$, f' is differentiable on $[x_1, x_3]$ and hence f' is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2). Hence, by Rolle's Theorem, there exists $c \in (c_1, c_2) \subset (x_1, x_3)$ such that $f''(c) = 0$.

2. Assume that P has at least three distinct real roots say $x_1 < x_2 < x_3$. Then, by the previous question, there exists $c \in (x_1, x_3)$ such that $P''(c) = 0$. But this is impossible since:

$$\forall x \in \mathbb{R},\ P''(x) = 12x^2 + 2\alpha \geq 2\alpha > 0.$$

Exercise 5.

1. For $x \in [1/\mathrm{e}, +\infty)$,

$$f'(x) = \ln(x) + \frac{x}{x} = \ln(x) + 1.$$

2. Since

$$\forall x \in (1/\mathrm{e}, +\infty),\ f'(x) > 0$$

and f is continuous, we conclude that f is increasing hence injective.

Moreover, by the the continuity of f again, and by (a corollary of) the Intermediate Value Theorem, we know that

$$f\big([1/\mathrm{e}, +\infty)\big) = \left[f(1/e), \lim_{+\infty} f\right) = [-1/e, +\infty).$$

Hence f is also onto.

3. Since $f(1) = 0$ we have $f^{-1}(0) = 1$. Now f is differentiable at 1 and $f'(1) = 1 \neq 0$ so we conclude, by the Inverse Function Rule, that f^{-1} is differentiable at 0 and that

$$\left(f^{-1}\right)'(0) = \frac{1}{1} = 1.$$

4. Since $f(1/\mathrm{e}) = -1/\mathrm{e}$ we have $f^{-1}(-1/\mathrm{e}) = 1/\mathrm{e}$. Now $f'(1/\mathrm{e}) = 0$ so we conclude that f^{-1} is not differentiable at $-1/\mathrm{e}$.

Exercise 6.

1.

$$f(x) \underset{x\to 0^+}{\sim} \frac{x^\alpha}{1 \times x} = x^{\alpha-1}, \qquad f(x) \underset{x\to+\infty}{\sim} \frac{x^\alpha}{x^2\,\dfrac{\mathrm{e}^{x/2}}{2}} = 2x^{\alpha-2}\mathrm{e}^{x/2}.$$

2. From the equivalent of f at 0^+ we conclude:

$$\lim_{0^+} f = \begin{cases} 0 & \text{if } \alpha > 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

hence f possesses an extension by continuity $\tilde{f}$ at 0 and

$$\tilde{f}(0) = \begin{cases} 0 & \text{if } \alpha > 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

3. Let $x \in \mathbb{R}_+^*$. Then:

$$\frac{\tilde{f}(x) - \tilde{f}(0)}{x - 0} = \frac{f(x)}{x} \underset{x\to 0^+}{\sim} x^{\alpha-2} \underset{x\to 0^+}{\longrightarrow} \begin{cases} 0 & \text{if } \alpha > 2 \\ 1 & \text{if } \alpha = 2 \\ +\infty & \text{if } \alpha < 2 \end{cases}$$

hence $\tilde{f}$ is differentiable (on the right) at 0 if and only if $\alpha \geq 2$ and in this case

$$\tilde{f}_r'(0) = \begin{cases} 0 & \text{if } \alpha > 2 \\ 1 & \text{if } \alpha = 2 \end{cases}$$

4. a) We take $\sinh(x) \underset{x\to 0}{=} x + x^3/6 + o(x^3)$ and we obtain:

$$\begin{aligned} x - \left(1+x^2\right)\sinh(x) &\underset{x\to 0}{=} x - \left(1+x^2\right)\left(x + \frac{x^3}{6} + o(x^3)\right) \\ &\underset{x\to 0}{=} x - \left(x + \frac{7x^3}{6} + o(x^3)\right) \\ &\underset{x\to 0}{=} -\frac{7x^3}{6} + o(x^3). \end{aligned}$$

b) Hence

$$x - \left(1+x^2\right)\sinh(x) \underset{x\to 0}{\sim} -\frac{7x^3}{6}.$$

So $\lambda = -7/6$.

c) Let $x \in \mathbb{R}_+^*$. Then:

$$\begin{aligned} \frac{\tilde{f}(x) - \tilde{f}(0)}{x-0} &= \frac{f(x)-1}{x} \\ &= \frac{\dfrac{x}{\left(1+x^2\right)\sinh(x)} - 1}{x} \\ &= \frac{x - \left(1+x^2\right)\sinh(x)}{x\left(1+x^2\right)\sinh(x)} \\ &\underset{x\to 0^+}{\sim} \frac{-7x^3/6}{x^2} = -\frac{7}{6}x \underset{x\to 0^+}{\longrightarrow} 0. \end{aligned}$$

Hence $\tilde{f}$ is differentiable from the right at 0 and we have $\tilde{f}_r'(0) = 0$.

Exercise 7.

1. We use $u(x) = x$ and $v'(x) = \sin(x)$ so that $u'(x) = 1$ and we can take $v(x) = -\cos(x)$:

$$\begin{aligned} I &= \left[-x\cos(x)\right]_{x=0}^{x=\pi/3} + \int_0^{\pi/3} \cos(x)\,\mathrm{d}x \\ &= -\frac{\pi}{3}\cos\left(\frac{\pi}{3}\right) - 0 + \left[\sin(x)\right]_{x=0}^{x=\pi/3} \\ &= -\frac{\pi}{6} + \sin\left(\frac{\pi}{3}\right) - 0 \\ &= -\frac{\pi}{6} + \frac{\sqrt{3}}{2}. \end{aligned}$$

2. With $t = \cos(x)$ we have $\mathrm{d}t = -\sin(x)\,\mathrm{d}x$, and when $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$. Then:

$$\begin{aligned} J &= \int_1^{-1} -\frac{\mathrm{d}t}{1+t^2} \\ &= \int_{-1}^{1} \frac{\mathrm{d}t}{1+t^2} \\ &= \left[\arctan(t)\right]_{t=-1}^{t=1} \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \\ &= \frac{\pi}{2}. \end{aligned}$$