INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #3 — 2h

March 17, 2022

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed.

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The questions of this exercise are independent from each other.

1. Show that the following function is differentiable and determine an expression of its derivative:

$$f : \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ t & \longmapsto & t^{\sin(t)}. \end{array}$$

2. Determine the value of the following limit:

$$\ell = \lim_{x\to 0} \frac{e^{\sin(x)} - 1}{\cos(x) - 1} \arctan(x).$$

3. Let $\alpha \in \mathbb{R}$. Find the simplest equivalent as $x \longrightarrow +\infty$ of $g(x) = x^{-\alpha}\left(1 + x^{\alpha}\right)$.

4. Determine the 3rd order Taylor–Young expansion as $x \longrightarrow 0$ of $u(x) = e^x \ln(1+x)$, and deduce the value of the following limit:

$$\lim_{x\to 0} \frac{e^x \ln(1+x) - x - x^2/2}{x^3}.$$

Exercise 2. The goal of this exercise is to compute an approximation of $\cos(1/3)$.

1. Recall the N-th order Taylor–Lagrange formula for a function f on an interval $[a, b]$.

2. Show that there exists a polynomial function P with $\deg P \leq 5$ and a number $\alpha \in \mathbb{R}_+^*$ such that

$$\forall x \in (0, \pi/2],\ P(x) - \alpha x^6 < \cos(x) < P(x).$$

3. You're given:

$$P\left(\frac{1}{3}\right) = \frac{1836}{1944} = 0.944\overline{958847736625514403292181069}, \qquad \alpha\left(\frac{1}{3}\right)^6 = \frac{1}{524880} < \frac{1}{500000} = 2 \cdot 10^{-6},$$

   where the line over the digits means that these digits are repeated infinitely.

   Using these values, give an approximation of $\cos(1/3)$ correct to as many decimal places as possible.

Exercise 3.

1. Recall the Mean Value Theorem.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$\forall x \in \mathbb{R},\ f'(x) \neq 1.$$

   Show by contradiction and by applying the Mean Value Theorem that there exists at most one element $a \in \mathbb{R}$ such that $f(a) = a$.

Exercise 4. Let x_1, x_2, x_3 be three real numbers with $x_1 < x_2 < x_3$, let $f : [x_1, x_3] \longrightarrow \mathbb{R}$ be a function twice differentiable such that $f(x_1) = f(x_2) = f(x_3)$.

1. Show, by applying Rolle's Theorem several times (to f and to f') that there exists $c \in (x_1, x_3)$ such that $f''(c) = 0$.
2. Application: show that for all $\alpha \in \mathbb{R}_+^*$, $\beta, \gamma \in \mathbb{R}$, the following polynomial function has at most two distinct real roots:

$$P : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^4 + \alpha x^2 + \beta x + \gamma. \end{array}$$

Exercise 5. Let

$$f : \begin{array}{ccc} [1/\mathrm{e}, +\infty) & \longrightarrow & [-1/\mathrm{e}, +\infty) \\ x & \longmapsto & x \ln(x). \end{array}$$

You're given that f is well defined and differentiable.

1. Compute the derivative of f.
2. Show that f is a bijection.
3. Show that f^{-1} is differentiable at 0 and determine the value of $\left(f^{-1}\right)'(0)$.
4. Is f^{-1} differentiable at $-1/\mathrm{e}$?

Exercise 6. Let $\alpha \in [1, +\infty)$ and define

$$f : \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x^\alpha}{\left(1 + x^2\right) \sinh(x)} \end{array}$$

1. Give the simplest equivalent of f at 0^+ and at $+\infty$.
2. Show that f possesses an extension by continuity $\tilde{f}$ at 0, and determine the value of $\tilde{f}(0)$.
3. In the case $\alpha > 1$, is $\tilde{f}$ differentiable (from the right) at 0? if it is the case, determine the value of $\tilde{f}_r'(0)$.
4. From now on we assume that $\alpha = 1$. We study the differentiability of $\tilde{f}$ (from the right) at 0 in this case.
   a) Determine the third order Taylor–Young expansion of $x - \left(1 + x^2\right) \sinh(x)$ as $x \longrightarrow 0$.
   b) Deduce that
   $$x - \left(1 + x^2\right) \sinh(x) \underset{x \to 0}{\sim} \lambda x^3$$
   where $\lambda \in \mathbb{R}^*$ you will determine.
   c) Is $\tilde{f}$ differentiable (from the right) at 0? if it is the case, determine the value of $\tilde{f}_r'(0)$.

Exercise 7. The questions of this exercise are independent from each other.

1. Use an integration by parts to compute the value of the following integral:
$$I = \int_0^{\pi/3} x \sin(x) \, \mathrm{d}x.$$
2. Use the substitution $t = \cos(x)$ to compute the value of the following integral:
$$J = \int_0^{\pi} \frac{\sin(x)}{1 + \cos^2(x)} \, \mathrm{d}x.$$