INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #4

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Exercise 1. With

$$\ln(1+x) \underset{x\to 0}{=} x - \frac{x^2}{2} - \frac{x^3}{3} + o\bigl(x^3\bigr)$$

and a long division we obtain:

$$f(x) \underset{x\to 0}{=} 1 + \frac{x}{2} - \frac{x^2}{12} + o\bigl(x^2\bigr)$$

from which we deduce

$$\lim_{x\to 0} f(x) = 1.$$

Hence f possesses an extension by continuity at 0.
Since $\tilde{f}$ possesses a first order Taylor–Young expansion at 0 we conclude that $\tilde{f}$ is differentiable at 0 at $\tilde{f}'(0) = 1/2$.
The equation of the tangent line Δ to the graph of f at 0 is:

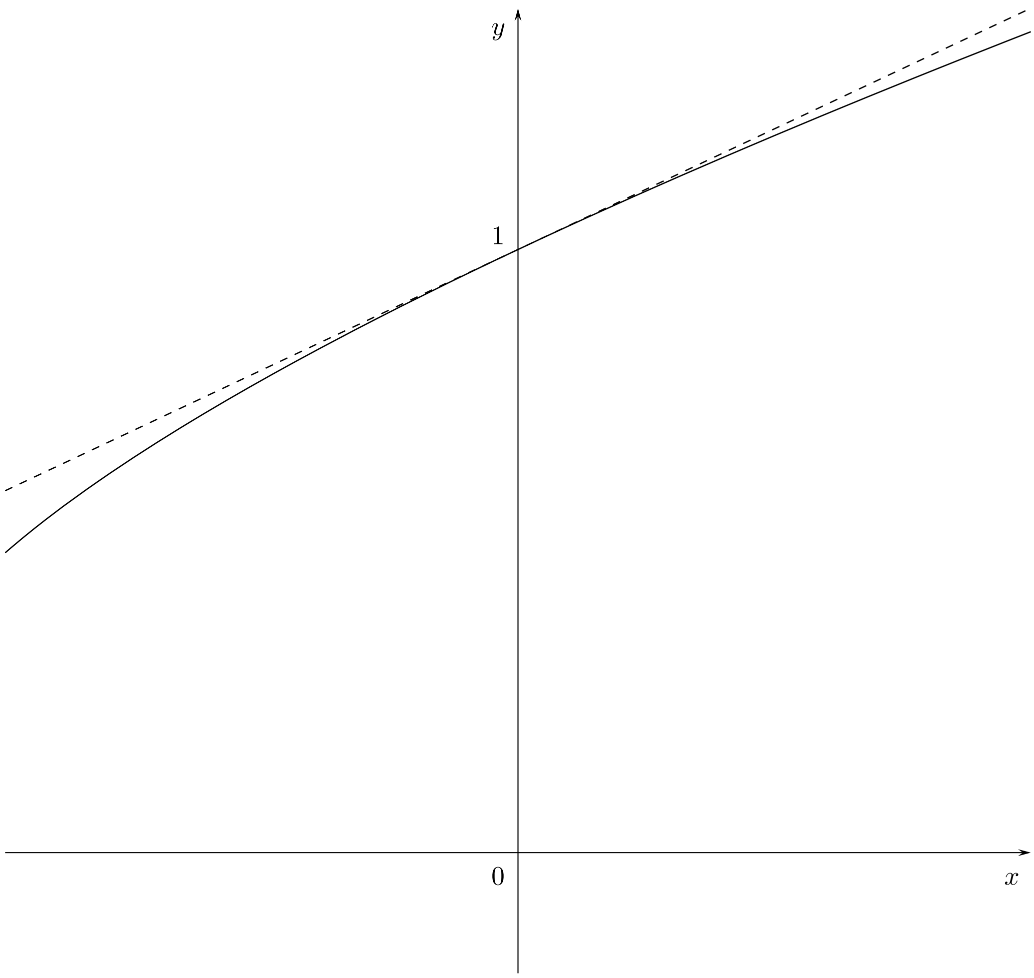
$$\Delta\colon y = 1 - \frac{x}{2}.$$

By looking at the sign of the second order term in the expansion:

$$-\frac{x^2}{12} < 0$$

we conclude that the graph of $\tilde{f}$ is below Δ in a neighborhood of 0.

Figure 5 – Graph of $\tilde{f}$ and Δ (dashed) in a neighborhood of 0 (Exercise 1)



Exercise 2.

1. Let $x \in \mathbb{R}_+^*$. Then $\left[x, \sqrt{x}\right] \subset \mathbb{R}_+^*$ where the function $t \mapsto \mathrm{e}^t/t^2$ is continuous. Hence

$$\int_x^{x+\sqrt{x}} \frac{\mathrm{e}^t}{t^2}\,\mathrm{d}t$$

is well defined.

2. Since the function $t \mapsto \mathrm{e}^t/t^2$ is continuous on $\mathbb{R}_+^*$ it possesses an antiderivative say F. Then, for all $x \in \mathbb{R}_+^*$,

$$G(x) = F\big(x + \sqrt{x}\big) - F(x)$$

and we conclude, by the Chain Rule, that G is differentiable and that:

$$\begin{aligned} G'(x) &= \left(1 + \frac{1}{2\sqrt{x}}\right) F'\big(x + \sqrt{x}\big) - F'(x) \\ &= \left(1 + \frac{1}{2\sqrt{x}}\right) \frac{\mathrm{e}^{x+\sqrt{x}}}{\big(x + \sqrt{x}\big)^2} - \frac{\mathrm{e}^x}{x^2} \end{aligned}$$

3. Let $x \in \mathbb{R}_+^*$. The function exp is continuous on $\left[x, x+\sqrt{x}\right]$ and the function $t \mapsto 1/t^2$ is (piecewise) continuous and positive on $\left[x, x+\sqrt{x}\right]$ hence, by the Mean Value Theorem (MVT2) there exists $c_x \in \left[x, x+\sqrt{x}\right]$ such that

$$G(x) = \mathrm{e}^{c_x} \int_x^{x+\sqrt{x}} \frac{\mathrm{d}t}{t^2} = \mathrm{e}^{c_x} \left[-\frac{1}{t}\right]_{t=x}^{t=x+\sqrt{x}} = \mathrm{e}^{c_x} \left(-\frac{1}{x+\sqrt{x}} + \frac{1}{x}\right) = \mathrm{e}^{c_x} \frac{\sqrt{x}}{x\big(x+\sqrt{x}\big)} = \frac{\mathrm{e}^{c_x}}{\sqrt{x}\big(x+\sqrt{x}\big)}$$

4. Let $x \in \mathbb{R}_+^*$. Since $c_x \in \left[x, x+\sqrt{x}\right]$, we conclude that

$$\mathrm{e}^x \le \mathrm{e}^{c_x} \le \mathrm{e}^{x+\sqrt{x}}$$

and hence $\lim\limits_{x \to 0^+} c_x = 1$. Moreover,

$$x + \sqrt{x} \underset{x \to 0^+}{\sim} \sqrt{x}.$$

We hence conclude that

$$G(x) \underset{x \to 0^+}{\sim} \frac{1}{x}.$$

and $\lim\limits_{x \to 0^+} G(x) = +\infty$.

From the inequality

$$\mathrm{e}^x \le \mathrm{e}^{c_x} \le \mathrm{e}^{x+\sqrt{x}}$$

we conclude

$$\frac{\mathrm{e}^x}{\sqrt{x}\big(x+\sqrt{x}\big)} \le G(x)$$

and since

$$\frac{\mathrm{e}^x}{\sqrt{x}\big(x+\sqrt{x}\big)} \xrightarrow[x \to +\infty]{} +\infty$$

(the exponential grows faster than the powers of x), we conclude that $\lim\limits_{x \to +\infty} G(x) = +\infty$.

Exercise 3.

1.

$$I_0 = \int_0^1 \mathrm{e}^x\,\mathrm{d}x = \mathrm{e} - 1.$$

2. Let $n \in \mathbb{N}$. For $x \in [0,1]$ we have:
$$0 \leq 1-x \leq 1$$
hence
$$0 \leq (1-x)^n \leq 1$$
and
$$0 \leq \mathrm{e}^x \leq \mathrm{e}$$
We hence conclude
$$0 \leq (1-x)^n \mathrm{e}^x \leq (1-x)^n \mathrm{e}.$$
Since $0<1$ we can integrate (with respect to x) this inequality:
$$0 \leq \int_0^1 (1-x)^n \mathrm{e}^x \,\mathrm{d}x \leq \int_0^1 (1-x)^n \mathrm{e} \,\mathrm{d}x.$$
Now
$$\int_0^1 (1-x)^n \,\mathrm{d}x = \left[\frac{1}{n+1}(1-x)^{n+1}\right]_{x=0}^{x=1} = \frac{1}{n+1}$$
and we conclude:
$$0 \leq I_n \leq \frac{1}{n!}\frac{\mathrm{e}}{n+1} = \frac{\mathrm{e}}{(n+1)!}.$$
By the Squeeze Theorem we conclude that $I_n \underset{n\to+\infty}{\longrightarrow} 0$.

3. Let $n \in \mathbb{N}$. By an integration by parts with $u(x) = (1-x)^{n+1}$ and $v'(x) = \mathrm{e}^x$ we have:
$$\begin{aligned}
I_{n+1} &= \frac{1}{(n+1)!}\int_0^1 (1-x)^{n+1}\mathrm{e}^x \,\mathrm{d}x \\
&= \frac{1}{(n+1)!}\left(\left[(1-x)^{n+1}\mathrm{e}^x\right]_{x=0}^{x=1} + (n+1)\int_0^1 (1-x)^n \mathrm{e}^x \,\mathrm{d}x\right) \\
&= \frac{1}{(n+1)!}\left(1 + (n+1)\int_0^1 (1-x)^n \mathrm{e}^x \,\mathrm{d}x\right) \\
&= \frac{1}{(n+1)!} + I_n
\end{aligned}$$

4. We can proceed by induction:
   - For $n=0$: from Question 1 we have
   $$I_0 = \mathrm{e} - 1 = \mathrm{e} - \sum_{k=0}^{0} \frac{1}{k}!$$
   - Assume that the result is true for some $n \in \mathbb{N}$. Then:
   $$\begin{aligned}
   I_{n+1} &= I_{n+1} - I_n + I_n \\
   &= -\frac{1}{(n+1)!} + I_n && \textit{by Question 3} \\
   &= -\frac{1}{(n+1)!} + \mathrm{e} - \sum_{k=0}^{n}\frac{1}{k!} && \textit{by the induction hypothesis} \\
   &= \mathrm{e} - \sum_{k=0}^{n+1}\frac{1}{k!}
   \end{aligned}$$

5. Since $I_n \underset{n\to+\infty}{\longrightarrow} 0$ we conclude
$$\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{k!} = \mathrm{e}.$$

Exercise 4.

- $0_E \in F$ since $0_E(2X) = 0_E = X0_E'(X)$. Hence $F \neq \emptyset$.
- Let $P, Q \in F$ and $\lambda \in \mathbb{R}$, and set $R = P + \lambda Q$. We now to show that $R \in F$:

$$R(2X) = P(2X) + \lambda Q(2X) = XP'(X) + \lambda XQ'(X) = XR'(X).$$

Hence F is a subspace of E.

Exercise 5.

1. Let $(x, y, z, t) \in E$. Then:

$$\begin{aligned}
(x,y,z,t) \in F \quad &\iff \quad \begin{cases} x - y + z - t = 0 \\ x + y - z - t = 0 \\ y - z = 0 \end{cases} \\
&\underset{R_2 \leftarrow R_2 - R_1}{\iff} \quad \begin{cases} x - y + z - t = 0 \\ 2y - 2z = 0 \\ y - z = 0 \end{cases} \\
&\iff \quad \begin{cases} x - y + z - t = 0 \\ y - z = 0 \end{cases} \\
&\iff \quad \begin{cases} x = t \\ y = z \\ z = z \\ t = t \end{cases} \\
&\iff \quad (x,y,z,t) = z(0,1,1,0) + t(1,0,0,1)
\end{aligned}$$

Hence a basis of F is:

$$\mathscr{B} = \big((0,1,1,0), (1,0,0,1)\big)$$

and $\dim F = 2$.

2.
- We first check that F and G are independent, i.e., $F \cap G = \{0_E\}$: let $w \in F \cap G$. Since $w \in G$, there exists $\alpha, \beta \in \mathbb{R}$ such that $w = \alpha u + \beta v = (\alpha + \beta, 0, \alpha - \beta, 0)$. Now since $w \in F$ we must have:

$$\begin{cases} (\alpha + \beta) - 0 + (\alpha - \beta) - 0 = 0 \\ (\alpha + \beta) + 0 - (\alpha - \beta) - 0 = 0 \\ 0 - (\alpha - \beta) = 0 \end{cases}$$

  from which we conclude $\alpha = \beta = 0$ and hence $w = 0_E$.
- Since u and v are not collinear, we know that $\dim G = 2$. By Grassmann's formula:

$$\dim(F \oplus G) = 2 + 2 = 4 = \dim E$$

We hence conclude, by the Inclusion–Equality Theorem, that $F \oplus G = E$.

Exercise 6.

1.
- $\operatorname{Ker} f$: let $(x, y, z) \in \mathbb{R}^3$. Then:

$$\begin{aligned}
(x,y,z) \in \operatorname{Ker} f \quad &\iff \quad f(x,y,z) = (0,0,0) \\
&\iff \quad \begin{cases} x + y + z = 0 \\ x - y - z = 0 \\ 3x + y + z = 0 \end{cases} \\
&\underset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 3R_1}}{\iff} \quad \begin{cases} x + y + z = 0 \\ -2y - 2z = 0 \\ -2y - 2z = 0 \end{cases}
\end{aligned}$$

$$\iff \quad \begin{cases} x + \ y + \ z = 0 \\ \quad -2y - 2z = 0 \end{cases} \quad \iff \quad \begin{cases} x = 0 y = -z z = z \end{cases}$$

$$\iff \quad (x, y, z) = z(0, -1, 1)$$

Hence $\operatorname{Ker} f = \operatorname{Span}\big\{(0, -1, 1)\big\}$ and a basis of $\operatorname{Ker} f$ is:

$$\big((0, -1, 1)\big)$$

- $\operatorname{Im} f$: we know that

$$\big(f(1, 0, 0), f(0, 1, 0), f(0, 0, 1)\big)$$

is a generating family of $\operatorname{Im} f$. Now,

$$f(1, 0, 0) = (1, 1, 3) \qquad f(0, 1, 0) = (1, -1, 1) \qquad f(0, 0, 1) = (1, -1, 1)$$

Since $f(0, 1, 0) = f(0, 0, 1)$ we conclude that a basis of $\operatorname{Im} f$ is:

$$\big((1, 1, 3), (1, -1, 1)\big).$$

Hence $\operatorname{rk} f = 2$.

2. Since $\operatorname{Ker} f \neq \{0_E\}$, f is not injective and since $\operatorname{Im} f \neq \mathbb{R}^3$, f is not surjective. f is not bijective.
3. Let E and F be two vector spaces over $\mathbb{K}$ and let $f\colon E \to F$ be a linear map. Then:

$$\dim E = \dim \operatorname{Ker} f + \operatorname{rk} f.$$

4. In our case, we have $E = F = \mathbb{R}^3$ and indeed:

$$3 = \dim E = \dim \operatorname{Ker} f + \operatorname{rk} f = 1 + 2.$$

Exercise 7.

1. Let $(x, y, z) \in E$ and let $\alpha, \beta, \gamma \in \mathbb{R}$. Then:

$$(x, y, z) = \alpha u_1 + \beta u_2 + \gamma u_3 \iff \begin{cases} \alpha + \beta = x \\ \alpha - \beta + \gamma = y \\ \beta + \gamma = z \end{cases}$$

$$\underset{R_2 \leftarrow R_2 - R_1}{\iff} \begin{cases} \alpha + \beta = x \\ -2\beta + \gamma = -x + y \\ \beta + \gamma = z \end{cases}$$

$$\underset{R_2 \leftrightarrow R_3}{\iff} \begin{cases} \alpha + \beta = x \\ \beta + \gamma = z \\ -2\beta + \gamma = -x + y \end{cases}$$

$$\underset{R_3 \leftarrow R_3 + 2R_2}{\iff} \begin{cases} \alpha + \beta = x \\ \beta + \gamma = z \\ 3\gamma = -x + y + 2z \end{cases}$$

$$\iff \begin{cases} \alpha = x - \beta = \dfrac{2x}{3} + \dfrac{y}{3} - \dfrac{z}{3} \\ \beta = z - \gamma = \dfrac{x}{3} - \dfrac{y}{3} + \dfrac{z}{3} \\ \gamma = -\dfrac{x}{3} + \dfrac{y}{3} + \dfrac{2z}{3} \end{cases}$$

Since this system possesses a unique solution, we conclude that $\mathscr{B}$ is a basis of E and for $v = (x, y, z) \in E$ one has:

$$[v]_{\mathscr{B}} = \begin{pmatrix} \dfrac{2x}{3} + \dfrac{y}{3} - \dfrac{z}{3} \\ \dfrac{x}{3} - \dfrac{y}{3} + \dfrac{z}{3} \\ -\dfrac{x}{3} + \dfrac{y}{3} + \dfrac{2z}{3} \end{pmatrix}$$

2. We know that a linear map is uniquely determined by the image of a basis of its domain. Since $\mathscr{B}$ is a basis of E we conclude that such an f exists and is unique.

3.
$$A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

4. From Question 1:
$$(1,0,0) = \frac{2}{3}u_1 + \frac{1}{3}u_2 - \frac{1}{3}u_3, \qquad (0,1,0) = \frac{1}{3}u_1 - \frac{1}{3}u_2 + \frac{1}{3}u_3, \qquad (0,0,1) = -\frac{1}{3}u_1 + \frac{1}{3}u_2 + \frac{2}{3}u_3,$$
hence
$$\begin{aligned} f(1,0,0) &= \frac{2}{3}f(u_1) + \frac{1}{3}f(u_2) - \frac{1}{3}f(u_3) \\ &= \frac{2}{3}(1,2) + \frac{1}{3}(-1,1) - \frac{1}{3}(-2,1) \\ &= \frac{1}{3}(3,4) \\ f(0,1,0) &= \frac{1}{3}f(u_1) - \frac{1}{3}f(u_2) + \frac{1}{3}f(u_3) \\ &= \frac{1}{3}(1,2) - \frac{1}{3}(-1,1) + \frac{1}{3}(-2,1) \\ &= \frac{1}{3}(0,2) \\ f(0,0,1) &= -\frac{1}{3}f(u_1) + \frac{1}{3}f(u_2) + \frac{2}{3}f(u_3) \\ &= -\frac{1}{3}(1,2) + \frac{1}{3}(-1,1) + \frac{2}{3}(-2,1) \\ &= \frac{1}{3}(-6,1) \end{aligned}$$
hence
$$B = \begin{pmatrix} 1 & 0 & -2 \\ 4/3 & 2/3 & 1/3 \end{pmatrix}$$

5. With
$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$
we have $A = BP$.