SCAN 1 — Solution of Math Test #1

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Exercise 1.

1. a) For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

b)

$$\sum_{k=0}^{n} q^n = \begin{cases} n+1 & \text{if } q = 1 \\ \dfrac{1-q^{n+1}}{1-q} & \text{otherwise} \end{cases}$$

2. Let $k \in \mathbb{N}$. By the Binomial Theorem we have:

$$\sum_{j=0}^{k} \binom{k}{j} x^{2j} = \sum_{j=0}^{k} \binom{k}{j} \left(x^2\right)^j 1^{k-j} = \left(1+x^2\right)^k$$

In particular, this sum is not nil, hence S is well defined, and:

$$S = \sum_{k=0}^{n} \frac{1}{\left(1+x^2\right)^k} = \sum_{k=0}^{n} \left(\frac{1}{1+x^2}\right)^k.$$

There are two cases:

- if $x = 0$, then $q = 1/\left(1+x^2\right) = 1$ and we have $S = n+1$,
- if $x \neq 0$, then $q = 1/\left(1+x^2\right) \neq 1$ and we have:

$$S = \frac{1-q^{n+1}}{1-q} = \frac{1 - \dfrac{1}{(1+x^2)^{n+1}}}{1 - \dfrac{1}{1+x^2}} = \frac{1}{x^2} + 1 + \frac{1}{x^2}\left(\frac{1}{1+x^2}\right)^n.$$

Exercise 2.

1. We know that $\sin(\pi/6) = 1/2 \geq 0$ (and also that $\pi/6 \geq 0$) hence, from $(*)$:

$$\frac{1}{2} \leq \frac{\pi}{6}$$

hence $\pi \geq 3$.

2.

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}-1\right)$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}+1\right)$$

$$\tan\left(\frac{\pi}{12}\right) = \frac{\sin(\pi/2)}{\cos(\pi/12)} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

3. From $(*)$ (with $\sin(\pi/12) \geq 0$ and $\pi/12 \geq 0$):

$$\sin\left(\frac{\pi}{12}\right) \leq \frac{\pi}{12}$$

hence

$$\frac{1}{2\sqrt{2}}\left(\sqrt{3}-1\right) \leq \frac{\pi}{12}$$

hence

$$\pi \geq \frac{12}{2\sqrt{2}}\left(\sqrt{3}-1\right) = 3\sqrt{2}\left(\sqrt{3}-1\right)$$

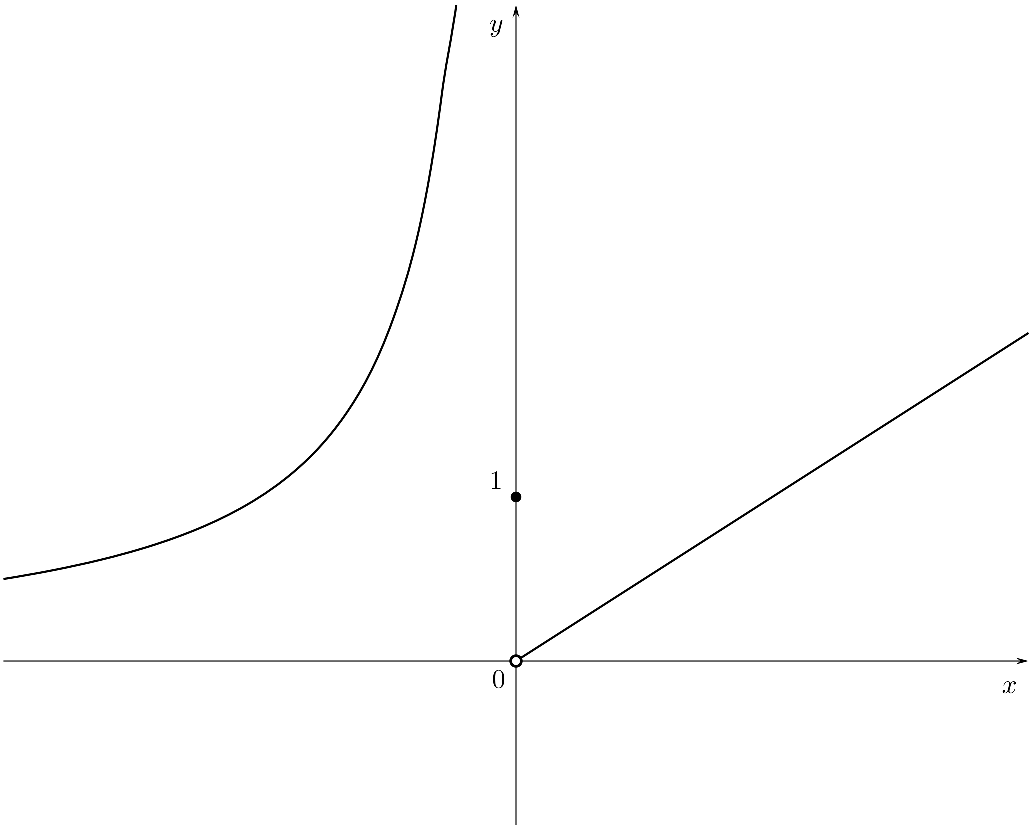

Figure 1 – Graph of function f of Exercise 3

Exercise 3.

1. See Figure 1.

2. The range of f is $f(\mathbb{R}) = \mathbb{R}_+^*$.

3.
   - f is not surjective since its range is $\mathbb{R}_+^*$ which is different from its codomain which is $\mathbb{R}$.
   - f is not injective since $f(-1) = -1/(-1) = 1 = f(1)$ yet $-1 \neq 1$.
   - f is not bijective (since not surjective).

4.

$$f(\mathbb{R}_+) = \mathbb{R}_+^*, \qquad f^{[-1]}(\mathbb{R}_-) = \emptyset,$$
$$f\big(\mathbb{R}_+^*\big) = \mathbb{R}_+^*, \qquad f^{[-1]}\big([1,+\infty)\big) = [-1,0] \cup [1,+\infty),$$
$$f\big([0,1)\big) = (0,1], \qquad f^{[-1]}\big((-\infty,1]\big) = (-\infty,-1] \cup [0,1],$$
$$f\big((-1,0]\big) = [1,+\infty), \qquad f^{[-1]}\big([1,2)\big) = \left[-1,-\frac{1}{2}\right] \cup \{0\} \cup [1,2).$$

Exercise 4.

1. Since the codomain of f is equal to (hence included in) its domain, $f \circ f$ is well-defined.

2.
   - f is injective: let $x, y \in A$ such that $f(x) = f(y)$. Then (applying f) yields

$$(f \circ f)(x) = (f \circ f)(y)$$

   i.e., $g(x) = g(y)$. Since g is injective, we conclude that $x = y$.
   Hence f is injective.

- f is surjective: let $y \in A$. Since g is surjective, there exists $a \in A$ such that $g(a) = y$. Define $x = f(a)$. We then have:
$$f(x) = f\big(f(a)\big) = g(a) = y.$$
Hence there exists $x \in A$ such that $f(y) = x$.
Hence f is surjective.

Hence f is a bijection.

Exercise 5.

1. Let $x \in \mathbb{R}$. In order for $f(x-1)+2$ to be well defined, we need $x - 1 \in \mathbb{R} \setminus \{-1\}$, i.e., $x \in \mathbb{R}^*$. Then $f(x-1)+2 \in \mathbb{R}$ as required (for the codomain of g). Conclusion: $D_g = \mathbb{R}^*$.

   We now simplify g: for $x \in \mathbb{R}^*$,
$$\begin{aligned} g(x) = f(x-1)+2 &= \frac{2\big(x-1\big)^2 + 2(x-1)+1}{x} + 2 = \frac{2x^2-4x+2+2x-2+1}{x} + 2 \\ &= \frac{2x^2-2x+1}{x} + 2 = \frac{2x^2+1}{x} \\ &= 2x + \frac{1}{x}. \end{aligned}$$
   The domain of g is symmetric with respect to the origin. We now show g is odd: let $x \in \mathbb{R}^*$. Then
$$g(-x) = -2x - \frac{1}{x} = -g(x)$$
   hence g is odd.

2. - g is increasing on $\big[1/\sqrt{2}, +\infty\big)$: let $x, y \in \big[1/\sqrt{2}, +\infty\big)$ such that $x < y$. Then, by the given hint:
$$g(x) - g(y) = \frac{(2xy-1)(x-y)}{xy}$$
     We know that $1/\sqrt{2} \leq x < y$, hence $1/2 < xy$ hence $(2xy-1) > 0$. Finally, since $x - y < 0$ and $xy > 0$ we conclude that $g(x) - g(y) < 0$ hence $g(x) < g(y)$.
   - g is decreasing on $\big(0, 1/\sqrt{2}\big]$: let $x, y \in \big(0, 1/\sqrt{2}\big]$ such that $x < y$. This time we have $0 < x < y \leq 1/\sqrt{2}$ so that $xy < 1/2$ and $(2xy-1) < 0$. We still have $x - y < 0$ and $xy > 0$ so that $g(x) - g(y) > 0$ hence $g(x) > g(y)$.

3. Notice that
$$\forall x \in \mathbb{R} \setminus \{-1\},\ f(x) = g(x+1) - 2$$
   so that the graph of f is obtained from that of g by a horizontal translation by 1 (to the left), and by a vertical translation by 2 (downward). See Figure 2.

Exercise 6.

- (P_1) is false: $x = y = 1/2 \in A$ but $x + y = 1 \notin A$.
- (P_2) is true: let $x \in A$. We define $y = (1-x)/2$. We show that $y \in A$ and that $x + y \in A$: since $0 < x < 1$ we conclude that $0 < 1 - x < 1$ and hence $0 < (1-x)/2 < 1/2 < 1$ so that $y \in A$. Now $x + y = (1+x)/2$. Since $0 < x < 1$ we conclude that $1/2 < (1+x)/2 < 1$ hence $(1+x)/2 \in A$.
- (P_3) is false: its negation is:

  $\neg(P_3)$ $$\forall x \in A,\ \exists y \in A,\ x + y \notin A.$$

  which is true: let $x \in A$. Since $0 < x < 1$, we have $0 < 1 - x < 1$. Now set $y = 1 - x \in A$, and we obtain $x + y = 1 \notin A$.

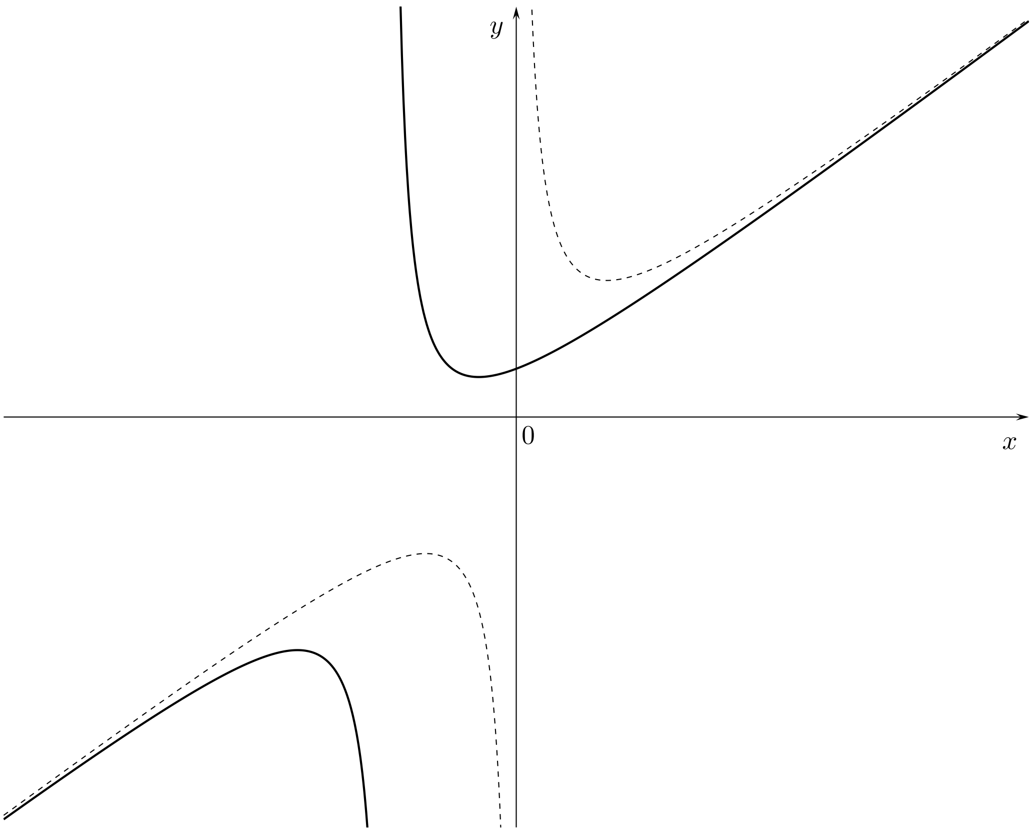

Figure 2 – Graph of the functions f and g (dashed) of Exercise 5. The graph of f is obtained from that of g by a horizontal translation to the left by 1 and a vertical translation, downward, by 2.