INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #2

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Exercise 1.

1. Since P has real coefficients,

$$1+i \text{ is a root of } P \iff 1-i \text{ is a root of } P,$$

we're hence going to perform the long division of P by Q where

$$\begin{aligned} Q\ :\ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto (x-1-i)(x-1+i)(x-1) = x^3-3x^2+4x-2. \end{aligned}$$

$$\begin{array}{r|l} & \quad 2x^2-x-1 \\ \hline x^3-3x^2+4x-2 & \quad 2x^5-7x^4+10x^3-5x^2-2x+2 \\ & -\big(2x^5-6x^4+\ 8x^3-4x^2\big) \\ \cline{2-2} & \qquad -\ x^4+\ 2x^3-\ x^2-2x+2 \\ & -\big(\quad -\ x^4+\ 3x^3-4x^2+2x\big) \\ \cline{2-2} & \qquad\qquad -\ x^3+3x^2-4x+2 \\ & -\big(\qquad\quad -\ x^3+3x^2-4x+2\big) \\ \cline{2-2} & \qquad\qquad\qquad\qquad\qquad 0 \end{array}$$

Hence:

$$\forall x\in\mathbb{R},\ P(x) = \big(2x^2-x-1\big)Q(x)+0$$

The quadratic $2x^2-x-1$ has roots 1 and $-1/2$, and we hence conclude that the roots of P (with their multiplicities) are:

- $1+i$ and $1-i$ both of multiplicity 1,
- 1 of multiplicity 2,
- $-1/2$ of multiplicity 1.

(Note that we have 5 roots counted with their multiplicities, which is consistent with the degree of P).

2. We now deduce the factored form of P in $\mathbb{R}$ and in $\mathbb{C}$:

$$\begin{aligned} P(x) &= 2\big(x^2-2x+2\big)(x-1)^2(x+1/2) && \text{(in } \mathbb{R}\text{)} \\ P(x) &= 2\big(x-1-i)(x-1+i)(x-1)^2(x+1/2) && \text{(in } \mathbb{C}\text{).} \end{aligned}$$

Exercise 2.

1. Let $x\in\mathbb{R}$.

a) We know that tanh is defined on $\mathbb{R}$, and that the range of tanh is $(-1,1)$, hence $1-\tanh>0$, so that A is well-defined.

$$A = \frac{1+\tanh(x)}{1-\tanh(x)} = \frac{1+\dfrac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}}{1-\dfrac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}} = \frac{\mathrm{e}^x+\mathrm{e}^{-x}+\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}-\mathrm{e}^x+\mathrm{e}^{-x}} = \frac{2\mathrm{e}^x}{2\mathrm{e}^{-x}} = \mathrm{e}^{2x}.$$

b) Let $n\in\mathbb{N}$. Then:

$$\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right)^n = A^n = \big(\mathrm{e}^{2x}\big)^n = \mathrm{e}^{2nx}$$

If we replace x by nx in A we obtain:

$$\frac{1+\tanh(nx)}{1-\tanh(nx)} = \mathrm{e}^{2nx},$$

hence the result.

2. a) Let $x \in \mathbb{R}$. Then, since the domain of arccosh is $[1, +\infty)$:

$$B \text{ is well defined} \iff x \neq 0 \ \wedge \ \frac{1}{2}\left(x + \frac{1}{x}\right) \geq 1$$

We split the problem in two cases:

- If $x > 0$ then

$$\frac{1}{2}\left(x + \frac{1}{x}\right) \geq 1 \iff x^2 + 1 \geq 2x \iff x^2 - 2x + 1 \geq 0 \iff (x-1)^2 \geq 0$$

  and we know that $(x-1)^2 \geq 0$ hence B is well defined in this case.
- If $x < 0$ then $x + 1/x < 0$ so that $(x + 1/x)/2 \notin [1, +\infty)$ hence B is not defined in this case.

Conclusion: $J = \mathbb{R}_+^*$.

b) Let $x \in J$. Define $y = \ln(x)$ so that $x = \mathrm{e}^y$. Then:

$$B = \operatorname{arccosh}\left(\frac{1}{2}\big(\mathrm{e}^y + \mathrm{e}^{-y}\big)\right) = \operatorname{arccosh}\big(\cosh(y)\big) = |y| = \big|\ln(x)\big|.$$

Exercise 3.

1.
   - $P_1 \neq \emptyset$: $0_E(0) + 0_E(1) = 0 + 0 = 0$ hence $0_E \in P_1$.
   - Let $P, Q \in P_1$ and let $\lambda \in \mathbb{R}$. Set $R = \lambda P + Q$. Then:

$$R(0) + R(1) = \lambda P(0) + Q(0) + \lambda P(1) + Q(1) = \lambda\big(P(0) + P(1)\big) + Q(0) + Q(1) = \lambda \times 0 + 0 = 0$$

   hence $R \in P_1$.

   Hence P_1 is a subspace of E.

2. Yes: we know that the intersection of subspaces of E is a subspace of E.

3. Let $P \in P_1 \cap P_2$. Since $P \in P_2$, there exists $a, b, c \in \mathbb{R}$ such that $P = a + bX + cX^2$. Since $P \in P_1$ we must have

$$P(0) + P(1) = a + a + b + c = 2a + b + c = 0.$$

Now,

$$\big\{2a + b + c = 0 \iff \begin{cases} a = -\frac{1}{2}b - \frac{1}{2}c \\ b = b \\ c = c \end{cases} \iff (a, b, c) = b\left(-\frac{1}{2}, 1, 0\right) + c\left(-\frac{1}{2}, 0, 1\right)$$

$$\iff P = b\left(-\frac{1}{2} + X\right) + c\left(-\frac{1}{2} + X^2\right)$$

We hence set:

$$p = -\frac{1}{2} + X \qquad \text{and} \qquad q = -\frac{1}{2} + X^2$$

and we have $P_1 \cap P_2 = \operatorname{Span}\{p, q\}$.

Exercise 4.

1. To determine the rank of (S) we perform the descent of the Gaussian elimination:

$$\begin{cases} x + y + z = m \\ x + 2y + 3z = 1 \\ 2x + 5y + 8z = 2 \end{cases} \underset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1}}{\iff} \begin{cases} x + y + z = m \\ y + 2z = 1 - m \\ 3y + 6z = 2 - 2m \end{cases} \underset{R_3 \leftarrow R_3 - 3R_2}{\iff} \begin{cases} x + y + z = m \\ y + 2z = 1 - m \\ 0 = -1 + m \end{cases}$$

Hence the rank of (S) is 2.

We now conclude that system (S) possesses solutions if and only if $m = 1$, in which case:

$$(S) \iff \begin{cases} x + y + z = 1 \\ y + 2z = 0 \end{cases} \iff \begin{cases} x = 1 - y - z = 1 + z \\ y = -2z \\ z = z \end{cases} \iff (x, y, z) = (1, 0, 0) + z(1, -2, 1).$$

2. a)

$$a \in \text{Span}\{u, v, w\} \iff \exists x, y, z \in \mathbb{R},\ a = xu + yv + zw \iff (x, y, z) \text{ are solutions of System } (S)$$

By Question 1 we conclude that

$$a \in \text{Span}\{u, v, w\} \iff m = 1.$$

b) Since $(0, 1, 2) \notin \text{Span}\{u, v, w\}$ (this is the vector a in the case $m = 0 \neq 1$) we conclude that $\text{Span}\{u, v, w\} \neq \mathbb{R}^3$, hence (u, v, w) is not a generating family of $\mathbb{R}^3$.

c) i) System (S') is just System (S) with nil constant terms (i.e., System (S') is the associated homogeneous system), $\text{rk}(S') = \text{rk}(S) = 2$.

ii) Since System (S') is homogeneous, it is compatible (hence possesses solutions, at least the nil solution); moreover it has 3 unknowns and is of rank 2, hence there's an infinite number of solutions that can be expressed in terms of 1 parameter.

iii) Checking whether (u, v, w) is an independent family reads as: let $x, y, z \in \mathbb{R}^3$ such that $xu + yv + zw = 0_E$. By writing the system associated to this equation yields System (S') which we know possesses other solutions than $(x, y, z) = (0, 0, 0)$, hence (u, v, w) is not an independent family.

Exercise 5.

1. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

$$\alpha a + \beta b = \gamma c + \delta d \iff \alpha a + \beta b - \gamma c - \delta d = 0_E \iff \begin{cases} \alpha + \beta - \gamma - \delta = 0 \\ \alpha + \beta + \gamma - \delta = 0 \\ \alpha \qquad\quad - \delta = 0 \\ \quad\ \beta \qquad - \delta = 0 \end{cases}$$

$$\underset{\substack{R_1 \leftrightarrow R_4 \\ \beta \text{ in 1st position}}}{\iff} \begin{cases} \beta \qquad\quad - \delta = 0 \\ \beta + \alpha + \gamma - \delta = 0 \\ \quad \alpha \qquad - \delta = 0 \\ \beta + \alpha - \gamma - \delta = 0 \end{cases} \underset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_4 \leftarrow R_4 - R_1}}{\iff} \begin{cases} \beta \qquad\quad - \delta = 0 \\ \quad \alpha + \gamma \qquad = 0 \\ \quad \alpha \qquad - \delta = 0 \\ \quad \alpha - \gamma \qquad = 0 \end{cases}$$

$$\underset{\substack{R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2}}{\iff} \begin{cases} \beta \qquad\quad - \delta = 0 \\ \quad \alpha + \ \gamma \qquad = 0 \\ \qquad - \ \gamma - \delta = 0 \\ \qquad - 2\gamma \qquad = 0 \end{cases} \iff \gamma = \alpha = \delta = \beta = 0.$$

Let $u \in F \cap G$. Since $u \in F = \text{Span}\{a, b\}$, there exists $\alpha, \beta \in \mathbb{R}$ such that $u = \alpha a + \beta b$. Since $u \in G = \text{Span}\{c, d\}$, there exists $\gamma, \delta \in \mathbb{R}$ such that $u = \gamma c + \delta d$. Hence we must have

$$\alpha a + \beta b = \gamma c + \delta d$$

hence $\alpha = \beta = 0$ which yields $u = \alpha a + \beta b = 0a + 0b = 0_E$. We conclude that $F \cap G = \{0_E\}$.

2. a) With $x, y, z, t, \alpha, \beta, \gamma, \delta$ as given,

$$u = \alpha a + \beta b + \gamma c + \delta d \iff \begin{cases} \alpha + \beta + \gamma + \delta = x \\ \alpha + \beta - \gamma + \delta = y \\ \alpha \qquad\quad + \delta = z \\ \quad\ \beta \qquad + \delta = t \end{cases}$$

$$\underset{\substack{R_1 \leftrightarrow R_4 \\ \beta \text{ in 1st position}}}{\iff} \begin{cases} \beta \qquad\quad + \delta = t \\ \beta + \alpha - \gamma + \delta = y \\ \quad \alpha \qquad + \delta = z \\ \beta + \alpha + \gamma + \delta = x \end{cases}$$

$$\underset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_4 \leftarrow R_4 - R_1}}{\iff} \begin{cases} \beta \qquad\quad + \delta = t \\ \quad \alpha - \gamma \qquad = y - t \\ \quad \alpha \qquad + \delta = z \\ \quad \alpha + \gamma \qquad = x - t \end{cases}$$

$$\underset{\substack{R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2}}{\iff} \begin{cases} \beta \qquad\quad + \delta = t \\ \quad \alpha - \ \gamma \qquad = y - t \\ \qquad\quad \gamma + \delta = z - y + t \\ \qquad 2\gamma \qquad = x - y \end{cases}$$

$$\iff \quad \begin{cases} \beta = t - \delta = \frac{1}{2}x + \frac{1}{2}y - z \\ \alpha = y - t + \gamma = \frac{1}{2}x + \frac{1}{2}y - t \\ \delta = z - y + t - \gamma = -\frac{1}{2}x - \frac{1}{2}y + z + t \\ \gamma = \frac{1}{2}x - \frac{1}{2}y \end{cases}$$

We conclude that there's a unique solution, namely:

$$\alpha = \frac{1}{2}x + \frac{1}{2}y - t \qquad \beta = \frac{1}{2}x + \frac{1}{2}y - z \qquad \gamma = \frac{1}{2}x - \frac{1}{2}y \qquad \delta = -\frac{1}{2}x - \frac{1}{2}y + z + t$$

We showed that $\mathrm{Span}\{a, b, c, d\} = E$, hence that (a, b, c, d) is a generating family of E.

b) Let $u = (x, y, z, t)$ and let $\alpha, \beta, \gamma, \delta$ as in the previous question. We know that $u = \alpha a + \beta b + \gamma c + \delta d$, and since $F = \mathrm{Span}\{a, b\}$ and $G = \mathrm{Span}\{c, d\}$ we set:

$$u_F = \alpha a + \beta b \in F, \qquad u_G = \gamma c + \delta d \in G$$

and we indeed have $u = u_F + u_G$. More precisely, the values of u_F and u_G are:

$$\begin{aligned} u_F &= \left(\frac{x+y}{2} - t, \frac{x+y}{2} - t, \frac{x+y}{2} - t, 0\right) + \left(\frac{x+y}{2} - z, \frac{x+y}{2} - z, 0, \frac{x+y}{2} - z\right) \\ &= \left(x + y - z - t, x + y - z - t, \frac{x+y}{2} - t, \frac{x+y}{2} - z\right) \\ u_G &= \left(\frac{x-y}{2}, -\frac{x-y}{2}, 0, 0\right) + \left(-\frac{x+y}{2} + z + t, -\frac{x+y}{2} + z + t, -\frac{x+y}{2} + z + t, -\frac{x+y}{2} + z + t\right) \\ &= \left(-y + z + t, -x + z + t, -\frac{x+y}{2} + z + t - \frac{x+y}{2} + z + t\right). \end{aligned}$$

c) We know that $F + G \subset E$. The inclusion $E \subset F + G$ is a direct consequence of Question b: let $u \in E$. By Question b there exists $u_F \in F$ and $u_G \in G$ such that $u = u_F + u_G$, hence $u \in F + G$. We conclude that $E = F + G$.

3. Yes: the direct sum $F \oplus G$ is valid since F and G are independent (Question 1) and from Question 2 we know that $E = F + G$. Hence $E = F \oplus G$.