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INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #3

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Exercise 1. 2.5

1.

$$\sup A = 1, \qquad \inf A = 0, \qquad \max A = 1, \qquad \min A \quad \text{DNE.}$$

2. See Figure 3.

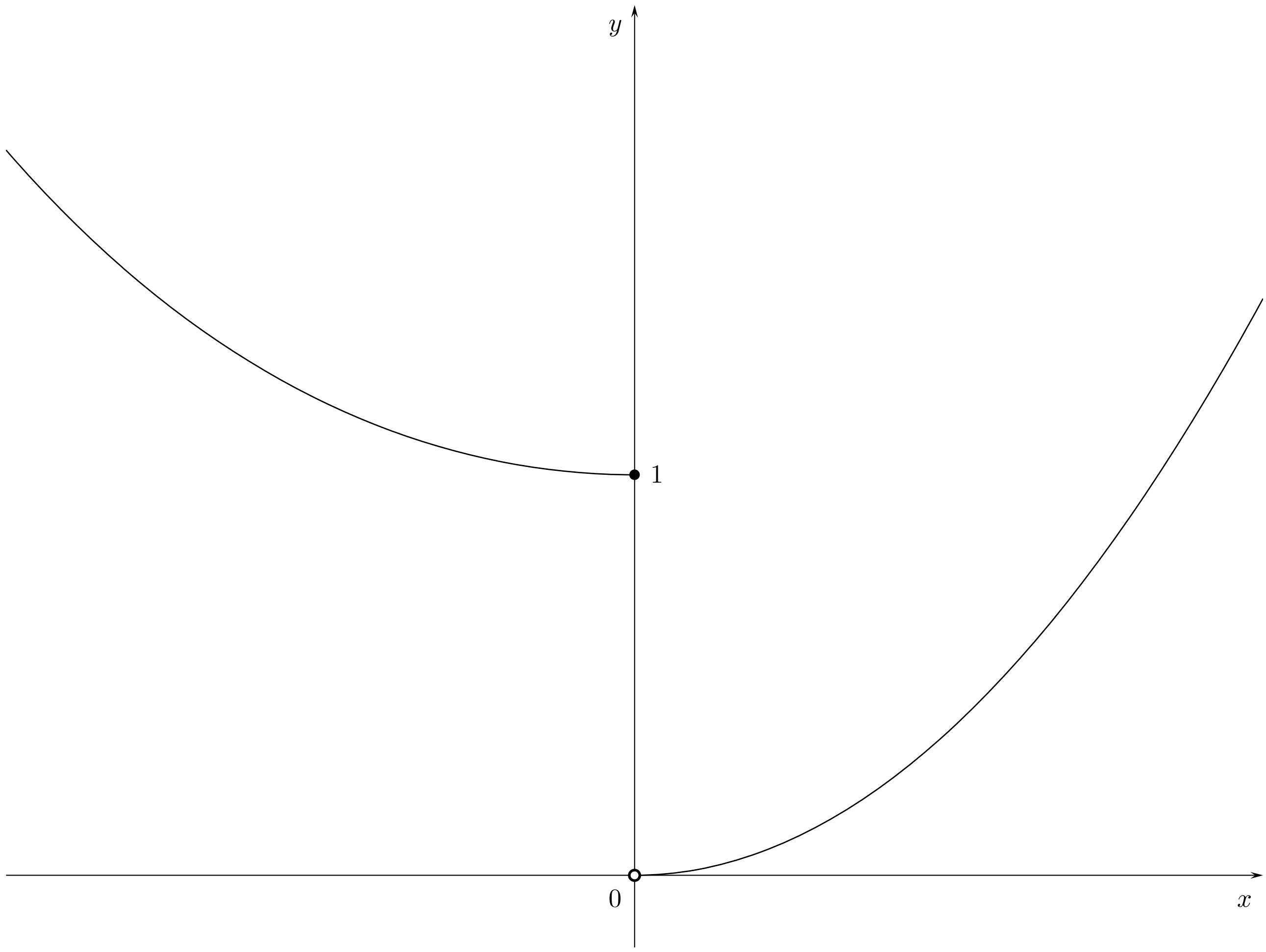

Figure 3 – Graph of the function f of Exercise 1.

$$\sup f = +\infty, \qquad \inf f = 0, \qquad \min f \quad \text{DNE}, \qquad \max f \quad \text{DNE},$$

Exercise 2. 6

1. (1) $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1.$

   (2) Let $x \in (-1,0)\cup(0,+\infty)$ (which is a punctured neighborhood of 0). Then:

$$\frac{\mathrm{e}^x - 1}{\ln(1+x)} = \frac{\mathrm{e}^x-1}{x}\,\frac{x}{\ln(1+x)} \underset{x\to 0}{\longrightarrow} 1,$$

   by the known limits

$$\lim_{x\to 0}\frac{\mathrm{e}^x-1}{x} = 1, \qquad \lim_{x\to 0} \ln(1+x)x = 1.$$

(3) Let $x \in (-2\pi, 0) \cup (0, 2\pi)$ (which is a punctured neighborhood of 0). Then:

$$\frac{\sin(x^2)}{\cos(x)-1} = \frac{\sin(x^2)}{x^2}\frac{x^2}{\cos(x)-1} \underset{x\to 0}{\longrightarrow} -2.$$

by the known limit

$$\lim_{x\to 0}\frac{\cos(x)-1}{x^2} = -\frac{1}{2}.$$

2. There are three cases:

- $\alpha > 0$: let $x \in [1, +\infty)$ (which is a neighborhood of $+\infty$). Then:

$$\frac{x^\alpha + 1}{x^\alpha + \ln(x)} = \frac{1 + x^{-\alpha}}{1 + x^{-\alpha}\ln(x)}.$$

Since $\alpha > 0$, $\lim_{x\to+\infty} x^{-\alpha} = 0$, and $\lim_{x\to+\infty} x^{-\alpha}\ln(x) = 0$ so that

$$\lim_{x\to+\infty}\frac{x^\alpha + 1}{x^\alpha + \ln(x)} = 1.$$

- $\alpha = 0$: in this case we're just computing:

$$\lim_{x\to+\infty}\frac{2}{1+\ln(x)} = 0.$$

- $\alpha < 0$: since $\lim_{x\to+\infty} x^\alpha = 0$ we directly conclude:

$$\lim_{x\to+\infty}\frac{x^\alpha + 1}{x^\alpha + \ln(x)} = \frac{0+1}{0+(+\infty)} = 0.$$

3. Let $x \in \mathbb{R}$. Then

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} \underset{x\to+\infty}{\longrightarrow} \frac{1-0}{1+0} = 1.$$

Since $\lim_{x\to+\infty}\frac{\sinh(x)}{\cosh(x)} = 1$ we can conclude that $\sinh \underset{+\infty}{\sim} \cosh$.

Exercise 3. 2

1. Let $x, y \in \mathbb{R}$ such that $f(x) = x$ and $f(y) = y$. Then, from the property satisfied by f we conclude

$$\left|f(x) - f(y)\right| = |x-y| \leq \frac{1}{2}|x-y|$$

hence

$$\frac{1}{2}|x-y| \leq 0$$

from which we obtain $x - y = 0$ i.e., $x = y$.

2. Let $x \in \mathbb{R}$. Then, from the property satisfied by f,

$$\left|f(x) - f(a)\right| = \left|f(x) - a\right| \leq \frac{1}{2}|x-a|.$$

Hence, by the Squeeze Theorem we conclude that

$$\lim_{x\to a}\left|f(x) - f(a)\right| = 0$$

i.e.,

$$\lim_{x\to a} f(x) = f(a).$$

Exercise 4.

5.5

1. We compute the rank of $\mathscr{B}$:

$$\operatorname{rk}\mathscr{B} = \operatorname{rk}\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix} \underset{\substack{C_2\leftarrow C_1+C_2\\ C_3\leftarrow C_3+C_1}}{=} \operatorname{rk}\begin{pmatrix}-1 & 0 & 0\\ 1 & 0 & 2\\ 1 & 2 & 0\end{pmatrix} = 3.$$

   Since $\#\mathscr{B} = \operatorname{rk}\mathscr{B} = \dim E = 3$ we conclude that $\mathscr{B}$ is a basis of E.

2. By definition, $P = P_1 + P_3 = 2X$.

3. We notice that

$$\frac{1}{2}P_1 + \frac{1}{2}P_2 = X^2, \qquad \frac{1}{2}P_1 + \frac{1}{2}P_3 = X, \qquad \frac{1}{2}P_2 + \frac{1}{2}P_3 = 1,$$

   so that

$$[X^2]_{\mathscr{B}} = \begin{pmatrix}1/2\\ 1/2\\ 0\end{pmatrix}, \qquad [X]_{\mathscr{B}} = \begin{pmatrix}1/2\\ 0\\ 1/2\end{pmatrix}, \qquad [1]_{\mathscr{B}} = \begin{pmatrix}0\\ 1/2\\ 1/2\end{pmatrix},$$

4. We rewrite Q as:

$$Q = a\left(\frac{1}{2}P_1 + \frac{1}{2}P_2\right) + b\left(\frac{1}{2}P_1 + \frac{1}{2}P_3\right) + c\left(\frac{1}{2}P_2 + \frac{1}{2}P_3\right) = \frac{a+b}{2}P_1 + \frac{a+c}{2}P_2 + \frac{b+c}{2}P_3$$

   so that we can take

$$Q_1 = \frac{a+b}{2}P_1 \in F_1, \qquad Q_2 = \frac{a+c}{2}P_2 + \frac{b+c}{2}P_3 \in F_2$$
$$= \frac{a+b}{2}(X^2+X-1) \qquad = \frac{a-b}{2}X^2 + \frac{-a+b}{2}X + \frac{a+b+2c}{2}.$$

5. a)
   - Since $0_E(0) = 0 = 0_E(1)$, we conclude that $0_E \in G$ hence that $G \neq \emptyset$.
   - Let $P, Q \in G$ and let $\lambda \in \mathbb{R}$, and set $R = P + \lambda Q$. We check that $R \in G$:

$$R(0) = P(0) + \lambda Q(0) = P(1) + \lambda Q(1) = R(1).$$

   b) No since $P = X \notin G$ since $P(0) = 0 \neq P(1) = 1$.

   c) Since $F_2 = \operatorname{Span}\{P_2, P_3\}$ and both F_2 and G are subspaces of E, we only need to check that $P_2 \in G$ and $P_3 \in G$: $P_2(0) = 1$ and $P_2(1) = 1$ so that $P_2 \in G$ and $P_3(0) = 1$ and $P_3(1) = 1$ so that $P_3 \in G$.

   Yes, we can in fact conclude that $F_2 = G$: since P_2 and P_3 are not collinear, the family (P_2, P_3) is independent hence $\dim F_2 = 2$. Now we know that $F_2 \subset G$, hence $\dim F_2 = 2 \leq \dim G$, and we also know that $G \neq E$, hence $\dim G < 3$ (hence $\dim G \leq 2$). Hence $\dim G = 2$. We finally conclude that $F_2 = G$ by the Inclusion–Equality Theorem.

   d) Since $\mathscr{B}$ is a basis of E, and from the definition of F_1 and F_2 we conclude that $E = F_1 \oplus F_2$. Since $F_2 = G$ we indeed have $E = F_1 \oplus G$.

Exercise 5.

3.5

1. Two possibilities (at least!):
   - Computing the rank of $\mathscr{B}$:

$$\operatorname{rk}\mathscr{B} = \operatorname{rk}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{pmatrix} \underset{C_3\leftarrow C_3-C_1}{=} \operatorname{rk}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 1 & 1 & -1\end{pmatrix} = 3,$$

     so that $\operatorname{rk}\mathscr{B} = \#\mathscr{B} = \dim E = 3$, hence $\mathscr{B}$ is a basis of E.

(quelle que soit la méthode)

- Solving a linear system: let $(x, y, z) \in E$ and $a, b, c \in \mathbb{R}$. Then:

$$au_1 + bu_2 + cu_3 = (x, y, z) \iff \begin{cases} a \quad + c = x \\ \quad b + c = y \\ a + b \quad = z \end{cases}$$

$$\underset{R_3 \leftarrow R_3 - R_1}{\iff} \begin{cases} a \quad + c = x \\ \quad b + c = y \\ \quad b - c = z - x \end{cases}$$

$$\underset{R_3 \leftarrow R_3 - R_2}{\iff} \begin{cases} a \quad + \quad c = x \\ \quad b + \quad c = y \\ \qquad - 2c = z - x - y \end{cases}$$

$$\iff \begin{cases} a = \quad \dfrac{1}{2}x - \dfrac{1}{2}y + \dfrac{1}{2}z \\ b = -\dfrac{1}{2}x + \dfrac{1}{2}y + \dfrac{1}{2}z \\ c = \quad \dfrac{1}{2}x + \dfrac{1}{2}y - \dfrac{1}{2}z \end{cases}$$

Since the system possesses a unique solution, we conclude that $\mathscr{B}$ is a basis of E (and as a byproduct we have the formula to express the coordinates of a vector of E in the basis $\mathscr{B}$).

2. We know that a linear map is uniquely determined by the image of a basis of its domain; since (u_1, u_2, u_3) is a basis of E (and since the values given are in F), we conclude that such an f exists and is unique.

3.
$$A = [f]_{\mathscr{B},\mathrm{std}_F} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. We need the values of $f(e_1)$, $f(e_2)$ and $f(e_3)$, and for this we need the coordinates of the vectors of e_1, e_2 and e_3 in the basis $\mathscr{B}$: using the result of the system solved in Question 1 (or by a direct computation) we have:

$$[e_1]_{\mathscr{B}} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \qquad [e_2]_{\mathscr{B}} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad [e_3]_{\mathscr{B}} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix},$$

i.e.,

$$e_1 = \frac{1}{2}u_1 - \frac{1}{2}u_2 + \frac{1}{2}u_3, \qquad e_2 = -\frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3, \qquad e_1 = \frac{1}{2}u_1 + \frac{1}{2}u_2 - \frac{1}{2}u_3,$$

so that (since f is linear),

$$\begin{aligned} f(e_1) &= \frac{1}{2}f(u_1) - \frac{1}{2}f(u_2) + \frac{1}{2}f(u_3) = \frac{1}{2}X - \frac{1}{2} + \frac{1}{2}\left(X^2 + 1\right) = \frac{1}{2}X^2 + \frac{1}{2}X + \frac{1}{2}, \\ f(e_2) &= -\frac{1}{2}f(u_1) + \frac{1}{2}f(u_2) + \frac{1}{2}f(u_3) = -\frac{1}{2}X + \frac{1}{2} + \frac{1}{2}\left(X^2 + 1\right) = \frac{1}{2}X^2 - \frac{1}{2}X + 1, \\ f(e_3) &= \frac{1}{2}f(u_1) + \frac{1}{2}f(u_2) - \frac{1}{2}f(u_3) = \frac{1}{2}X + \frac{1}{2} - \frac{1}{2}\left(X^2 + 1\right) = -\frac{1}{2}X^2 + \frac{1}{2}X, \end{aligned}$$

so that

$$[f]_{\mathscr{B}} = \begin{pmatrix} 1/2 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}.$$

5. We can compute the rank of f:

$$\begin{aligned} \operatorname{rk} f &= \operatorname{rk} A \\ &= \operatorname{rk} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$\mathrm{Im} f = F$

ou $\mathrm{Ker} f = \{0_E\}$.

$$\begin{aligned} &\underset{C_3 \leftarrow C_3 - C_2}{=} \quad \operatorname{rk}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \\ &= \quad 3 \end{aligned}$$

Hence $\operatorname{rk} f = \dim F = 3 < +\infty$ hence f is surjective, hence $\operatorname{Im} f = F$. Since $\dim E = \dim F < +\infty$ we know, from the Rank–Nullity Theorem, that

$$f \text{ is injective} \iff f \text{ is surjective} \iff f \text{ is bijective},$$

and we hence conclude that f is a bijection.

3.5

Exercise 6.

1. Let $(x,y,z) \in E$.

$$f(x,y,z) = (x, -x+3y-z, -x+2y), \qquad \begin{aligned} f_1(x,y,z) &= (x,y,z) - (x, -x+3y-z, -x+2y) \\ &= (0, x-2y+z, x-2y+z). \end{aligned}$$

2. Let $(x,y,z) \in E$. Then:

$$\begin{aligned} (x,y,z) \in \operatorname{Ker} f_1 &\iff f_1(x,y,z) = 0_E \\ &\iff x - 2y + z = 0 \\ &\iff \begin{cases} x = 2y - z \\ y = y \\ z = z \end{cases} \\ &\iff (x,y,z) = y(2,1,0) + z(-1,0,1), \end{aligned}$$

hence a basis of $\operatorname{Ker} f_1$ is: $\big((2,1,0),(-1,0,1)\big)$.

3. Let $\lambda \in \mathbb{R} \setminus \{1\}$. We now determine the rank of the system associated with $\operatorname{Ker} f_\lambda$: let $(x,y,z) \in E$. Then:

$$\begin{aligned} (x,y,z) \in \operatorname{Ker} f_\lambda \quad &\iff \quad f(x,y,z) - \lambda \cdot (x,y,z) = 0_E \\ &\iff \begin{cases} (1-\lambda)x & = 0 \\ -x + (3-\lambda)y - z & = 0 \\ -x + 2y - \lambda z & = 0 \end{cases} \\ &\underset{\substack{R_2 \leftarrow R_2 + \frac{1}{1-\lambda} R_1 \\ R_2 \leftarrow R_3 + \frac{1}{1-\lambda} R_1}}{\iff} \begin{cases} (1-\lambda)x & = 0 \\ (3-\lambda)y - z & = 0 \\ 2y - \lambda z & = 0 \end{cases} \qquad \textit{since } 1 - \lambda \neq 0 \\ &\underset{R_3 \leftarrow R_3 - \lambda R_2}{\iff} \begin{cases} (1-\lambda)x & = 0 \\ (3-\lambda)y - z & = 0 \\ \big(2 - 3\lambda + \lambda^2\big)y & = 0 \end{cases} \end{aligned}$$

so that this system is of rank < 3 if and only if $\lambda = 2$ (since $\lambda \neq 1$). In this case,

$$\begin{aligned} (x,y,z) \in \operatorname{Ker} f_2 \quad &\iff \quad \begin{cases} -x & = 0 \\ y - z & = 0 \end{cases} \\ &\iff \begin{cases} x = 0 \\ y = z \\ z = z \end{cases} \\ &\iff (x,y,z) = z(0,1,1). \end{aligned}$$

Hence a basis of $\operatorname{Ker} f_2$ is: $\big((0,1,1)\big)$.