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INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #4

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Exercise 1. 4.5

1. The square, the cosine and the exponential are all differentiable on $\mathbb{R}$, hence f is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R},\ f'(x) = 2x\cos\bigl(x^2\bigr)\mathrm{e}^{\cos(x^2)}.$$

2. - $D_g = [-1, 1]$.
   - arcsin is differentiable on $(-1, 1)$ and the absolute value function is differentiable on $\mathbb{R}^*$. Hence, by the Chain Rule, g is differentiable on $(-1, 0) \cup (0, 1)$ and

$$\begin{aligned}\forall x \in (-1,0)\cup(0,1),\ g'(x) &= \begin{cases} \arcsin(x) + \dfrac{x-1}{\sqrt{1-x^2}} & \text{if } x > 0 \\ -\arcsin(x) + \dfrac{-x-1}{\sqrt{1-x^2}} & \text{if } x < 0. \end{cases} \\ &= \bigl|\arcsin(x)\bigr| - \sqrt{\frac{1-|x|}{1+|x|}}.\end{aligned}$$

   - Differentiability of g at 0: let $x \in [-1, 0) \cup (0, 1]$:

$$\frac{g(x)-g(0)}{x} = \frac{g(x)}{x} = \frac{\arcsin(x)}{x}\bigl(|x|-1\bigr) \underset{x\to 0}{\sim} \frac{x}{x}(-1) = -1 \underset{x\to 0}{\longrightarrow} -1.$$

   Hence g is differentiable at 0 and $g'(0) = -1$.
   - Differentiability of g at 1 and -1: let $x \in [0, 1)$:

$$\frac{g(x)-g(1)}{x-1} = \frac{g(x)}{x} = \frac{(x-1)\arcsin(x)}{x-1} = \arcsin(x) \underset{x\to 1^-}{\longrightarrow} = \frac{\pi}{2}.$$

   Hence g is differentiable (from the left) at 1 and $g'_\ell(-1) = \pi/2$. Since g is odd, we conclude that g is differentiable (from the right) at -1 and (since g' is even), $g'_r(-1) = g'_\ell(1) = \pi/2$.

   Finally, we notice that

$$\forall x \in [-1, 1],\ g'(x) = \bigl|\arcsin(x)\bigr| - \sqrt{\frac{1-|x|}{1+|x|}}.$$

Exercise 2. 5.5

1. Let $x \in \mathbb{R}$. By the Chain Rule and the Product Rule:

$$\forall x \in \mathbb{R},\ f'(x) = \frac{x}{1+x^2} + \arctan(x) - \frac{x}{1+x^2} = \arctan(x).$$

2. a) Let $x \in \mathbb{R}_+^*$:

$$\ln\bigl(1+x^2\bigr) = \ln\left(x^2\left(1+\frac{1}{x^2}\right)\right) = \ln\bigl(x^2\bigr) + \ln\left(1+\frac{1}{x^2}\right) \underset{x\to+\infty}{=} 2\ln(x) + o(1) \underset{x\to+\infty}{\sim} 2\ln(x).$$

   Since $\dfrac{\ln(x)}{x} \underset{x\to+\infty}{\longrightarrow} 0$, we conclude that $2\ln(x) \underset{x\to+\infty}{=} o(x)$.

   b) Since $\lim_{+\infty} \arctan = \pi/2$, $x\arctan(x) \underset{x\to+\infty}{\sim} x\pi/2$.

   c)

$$f(x) = x\arctan(x) - \frac{1}{2}\ln\bigl(1+x^2\bigr) \underset{x\to+\infty}{=} x\arctan(x) + o(x).$$

   Since $x\arctan(x) \underset{x\to+\infty}{\sim} x\pi/2$,

$$f(x) \underset{x\to+\infty}{=} x\pi 2 + o(x) \underset{x\to+\infty}{\sim} \frac{\pi}{2}x.$$

   Hence $\lim\limits_{+\infty} f = +\infty$.

3. a) Since f' is positive on $\mathbb{R}_+^*$ we conclude that f is increasing on $\mathbb{R}_+$. Moreover, $f(0) = \arctan(0) = 0$, so that for all $x \in \mathbb{R}_+$, $f(x) + 1 \in [1, +\infty)$. Hence g is well defined.

Since f is increasing on $\mathbb{R}_+$, g is increasing hence injective.

Since $\lim\limits_{+\infty} f = +\infty$ and since f is continuous and increasing we conclude by (a corollary) of Bolzano's Theorem: $f(\mathbb{R}_+) = [1, +\infty)$, hence g is surjective.

b) We have:

- $\mathbb{R}_+^*$ is an interval,
- f is differentiable on $\mathbb{R}_+^*$,
- for $x \in \mathbb{R}_+^*$, $g'(x) = f'(x) = \arctan(x) \neq 0$,

hence, by the Inverse Function Rule, for all $x_0 \in \mathbb{R}_+^*$, g^{-1} is differentiable at $g(x_0)$ and

$$(g^{-1})'\big(g(x_0)\big) = \frac{1}{g'(x_0)} = \frac{1}{\arctan x_0}.$$

Or, equivalently, for all $y \in g(\mathbb{R}_+) = (1, +\infty)$, g^{-1} is differentiable at y and

$$(g^{-1})'(y) = \frac{1}{g'\big(g^{-1}(y)\big)} = \frac{1}{\arctan g^{-1}(y)}.$$

Exercise 3. 5

1. Let $x \in \mathbb{R}_+^*$. Then

$$f'(x) = \beta x^{\beta-1} \sin\left(\frac{1}{x}\right) - x^{\beta-2} \cos\left(\frac{1}{x}\right).$$

Since f is odd, f' is even and we can write:

$$\forall x \in \mathbb{R}^*, \;\; f'(x) = |x|^{\beta-2} \left(x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right).$$

2. Assume $\beta > 1$, and let $x \in \mathbb{R}^*$. Then

$$\frac{f(x) - f(0)}{x} = \frac{|x|^\beta}{x} \sin\left(\frac{1}{x}\right) \underset{x \to 0}{\longrightarrow} 0$$

since $\lim\limits_{x \to 0} \dfrac{|x|^\beta}{x} = 0$ and sin is bounded. Hence f is differentiable at 0 and $f'(0) = 0$.

3. Assume $\beta \leq 1$ and let $x \in \mathbb{R}_+^*$ (we will prove that f is not differentiable from the right at 0).

$$\frac{f(x) - f(0)}{x} = x^{\beta-1} \sin\left(\frac{1}{x}\right)$$

We prove that the limit $\lim\limits_{x \to 0^+} x^{\beta-1} \sin\left(\dfrac{1}{x}\right)$ doesn't exist in $\mathbb{R}$ by contradiction: assume

$$\lim_{x \to 0^+} x^{\beta-1} \sin\left(\frac{1}{x}\right) = \ell \in \mathbb{R}$$

Then, by the elementary operations on limits,

$$\lim_{+\infty} \sin = \lim_{x \to 0^+} x^{1-\beta} x^{\beta-1} \sin\left(\frac{1}{x}\right) = \begin{cases} \ell & \text{if } \beta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

which is impossible. Hence f is not differentiable from the right at 0, hence f is not differentiable at 0.

4. For f to be of class C^1 we need $\beta > 1$, but this is not enough, we also need $\lim\limits_{x \to 0} f'(x) = f'(0) = 0$.

- If $\beta > 2$ then $\lim\limits_{x \to 0} |x|^{\beta-2} = 0$, and since sin and cos are bounded, we have

$$\lim_{0} f' = \lim_{x \to 0} |x^{\beta-2}| \left(x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right) = 0 = f'(0)$$

so that f is of class C^1 in this cases.

- If $\beta \leq 2$, we show that $\lim 0^+ f'$ doesn't exist. Let $x \in \mathbb{R}_+$. Then

$$f'(x) = x^{\beta-2}\left(x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right).$$

We proceed by contradiction and assume that $\lim\limits_{0^+} f' = \ell \in \mathbb{R}$. Notice that

$$\cos\left(\frac{1}{x}\right) = x\sin\left(\frac{1}{x}\right) - x^{2-\beta}f'(x),$$

and that, since sin is bounded:

$$x\sin\left(\frac{1}{x}\right) \underset{x\to 0^+}{\longrightarrow} 0.$$

Then

$$\lim_{+\infty}\cos = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) - x^{2-\beta}f'(x) = \begin{cases} -\ell & \text{if } \beta = 2 \\ 0 & \text{otherwise} \end{cases}$$

which is impossible. Hence $\lim\limits_{0^+} f'$ doesn't exist, hence f is not of class C^1.

We conclude that f is of class C^1 if and only if $\beta > 2$.

5. For example the function

$$\begin{aligned} f \; : \; \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} |x|^{3/2}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \end{aligned}$$

This corresponds to the case $\beta = 3/2$: $\beta > 1$ so that f is differentiable, but $\beta \leq 2$ so that f is not of class C^1.

Exercise 4. 3

1.
$$\mathrm{e}^x \underset{x\to 0}{=} 1 + x + \frac{x^2}{2} + o(x^2), \qquad \sin(x)x \underset{x\to 0}{=} x - \frac{x^3}{6} + o(x^3).$$

2.
$$\begin{aligned} \mathrm{e}^x\sin(x) &\underset{x\to 0}{=} \left(1 + x + \frac{x^2}{2} + o(x^2)\right)\left(1 - \frac{x^3}{6} + o(x^3)\right) \\ &\underset{x\to 0}{=} x + x^2 + \frac{x^3}{3} + o(x^3). \end{aligned}$$

3.
$$\begin{aligned} f(x) &\underset{x\to 0}{=} x + o(x) \underset{x\to 0}{\sim} x, \\ f(x) - x &\underset{x\to 0}{=} x^2 + o(x^2) \underset{x\to 0}{\sim} x^2, \\ f(x) - x - x^2 &\underset{x\to 0}{=} \frac{x^3}{3} + o(x^3) \underset{x\to 0}{\sim} \frac{x^3}{3}. \end{aligned}$$

4.
$$\frac{1}{x^3}\left(\mathrm{e}^x\sin(x) - x - x^2\right) \underset{x\to 0}{\sim} \frac{1}{3} \underset{x\to 0}{\longrightarrow} \frac{1}{3}$$

hence $\ell = 1/3$.

Exercise 5. 2

1. See booklet.

2. See lecture.