

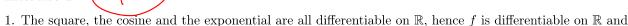


SCAN 1 — Solution of Math Test #4

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Exercise 1.



$$\forall x \in \mathbb{R}, \ f'(x) = 2x \cos(x^2) e^{\cos(x^2)}.$$

2. •
$$D_g = [-1, 1]$$
.

• arcsin is differentiable on (-1,1) and the absolute value function is differentiable on \mathbb{R}^* . Hence, by the Chain Rule, g is differentiable on $(-1,0) \cup (0,1)$ and

$$\forall x \in (-1,0) \cup (0,1), \ g'(x) = \begin{cases} \arcsin(x) + \frac{x-1}{\sqrt{1-x^2}} & \text{if } x > 0 \\ -\arcsin(x) + \frac{-x-1}{\sqrt{1-x^2}} & \text{if } x < 0. \end{cases}$$
$$= \left| \arcsin(x) \right| - \sqrt{\frac{1-|x|}{1+|x|}}.$$

• Differentiability of g at 0: let $x \in [-1, 0) \cup (0, 1]$:

$$\frac{g(x)-g(0)}{x} = \frac{g(x)}{x} = \frac{\arcsin(x)}{x} \left(|x|-1\right) \underset{x\to 0}{\sim} \frac{x}{x}(-1) = -1 \underset{x\to 0}{\longrightarrow} -1.$$

Hence g is differentiable at 0 and g'(0) = -1.

• Differentiability of g at 1 and -1: let $x \in [0,1)$:

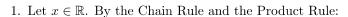
$$\frac{g(x)-g(1)}{x-1} = \frac{g(x)}{x} = \frac{(x-1)\arcsin(x)}{x-1} = \arcsin(x) \underset{x\to 1^-}{\longrightarrow} = \frac{\pi}{2}.$$

Hence g is differentiable (from the left) at 1 and $g'_{\ell}(-1) = \pi/2$. Since g is odd, we conclude that g is differentiable (from the right) at -1 and (since g' is even), $g'_r(-1) = g'_\ell(1) = \pi/2$.

Finally, we notice that

$$\forall x \in [-1, 1], \ g'(x) = \left| \arcsin(x) \right| - \sqrt{\frac{1 - |x|}{1 + |x|}}.$$

Exercise 2.



$$\forall x \in \mathbb{R}, \ f'(x) = \frac{x}{1+x^2} + \arctan(x) - \frac{x}{1+x^2} = \arctan(x).$$

2. a) Let $x \in \mathbb{R}^*_{\perp}$:

c)

$$\ln(1+x^2) = \ln\left(x^2\left(1+\frac{1}{x^2}\right)\right) = \ln(x^2) + \ln\left(1+\frac{1}{x^2}\right) \underset{x\to+\infty}{=} 2\ln(x) + o(1) \underset{x\to+\infty}{\sim} 2\ln(x).$$

Since $\frac{\ln(x)}{x} \xrightarrow[x \to +\infty]{} 0$, we conclude that $2\ln(x) = o(x)$.

 \Rightarrow b) Since $\lim +\infty \arctan = \pi/2$, $x \arctan(x) \sim x\pi/2$.

$$f(x) = x \arctan(x) - \frac{1}{2} \ln(1 + x^2) = x \arctan(x) + o(x).$$

Since $x \arctan(x) \underset{x \to +\infty}{\sim} x\pi/2$,

$$f(x) = x\pi 2 + o(x) \sim \frac{\pi}{x \to +\infty} x$$

Hence $\lim_{+\infty} f = +\infty$.

- 3. a) Since f' is positive on \mathbb{R}_+^* we conclude that f is increasing on \mathbb{R}_+ . Moreover, $f(0) = \arctan(0) = 0$, so that for all $x \in \mathbb{R}_+$, $f(x) + 1 \in [1, +\infty)$. Hence g is well defined.
 - Since f is increasing on \mathbb{R}_+ , g is increasing hence injective.
 - Since $\lim_{t\to\infty} f = +\infty$ and since f is continuous and increasing we conclude by (a corollary) of Bolzano's Theorem: $f(\mathbb{R}_+) = [1, +\infty)$, hence g is surjective.
 - b) We have:
 - \mathbb{R}_+^* is an interval,
 - f is differentiable on \mathbb{R}_+^* ,
 - for $x \in \mathbb{R}_+^*$, $g'(x) = f'(x) = \arctan(x) \neq 0$,

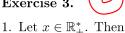
hence, by the Inverse Function Rule, for all $x_0 \in \mathbb{R}_+^*$, g^{-1} is differentiable at $g(x_0)$ and

$$(g^{-1})'(g(x_0)) = \frac{1}{g'(x_0)} = \frac{1}{\arctan x_0}.$$

Or, equivalently, for all $y \in g(\mathbb{R}_+) = (1, +\infty), g^{-1}$ is differentiable at y and

$$(g^{-1})'(y) = \frac{1}{g'(g^{-1}(y))} = \frac{1}{\arctan g^{-1}(y)}.$$

Exercise 3.



$$f'(x) = \beta x^{\beta - 1} \sin\left(\frac{1}{x}\right) - x^{\beta - 2} \cos\left(\frac{1}{x}\right).$$

Since f is odd, f' is even and we can write:

$$\forall x \in \mathbb{R}^*, \ f'(x) = |x|^{\beta - 2} \left(x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right).$$

2. Assume $\beta > 1$, and let $x \in \mathbb{R}^*$. Then

$$\frac{f(x) - f(0)}{x} = \frac{|x|^{\beta}}{x} \sin\left(\frac{1}{x}\right) \xrightarrow[x \to 0]{} 0$$

since $\lim_{x\to 0} \frac{|x|^{\beta}}{x} = 0$ and sin is bounded. Hence f is differentiable at 0 and f'(0) = 0.

3. Assume $\beta \leq 1$ and let $x \in \mathbb{R}_+^*$ (we will prove that f is not differentiable from the right at 0).

$$\frac{f(x) - f(0)}{x} = x^{\beta - 1} \sin\left(\frac{1}{x}\right)$$

We prove that the limit $\lim_{x\to 0^+} x^{\beta-1} \sin\left(\frac{1}{x}\right)$ doesn't exist in \mathbb{R} by contradiction: assume

$$\lim_{x \to 0^+} x^{\beta - 1} \sin\left(\frac{1}{x}\right) = \ell \in \mathbb{R}$$

Then, by the elementary operations on limits,

$$\lim_{+\infty}\sin=\lim_{x\to 0^+}x^{1-\beta}x^{\beta-1}\sin\left(\frac{1}{x}\right)=\begin{cases} \ell & \text{if }\beta=1\\ 0 & \text{otherwise}. \end{cases}$$

which is impossible. Hence f is not differentiable from the right at 0, hence f is not differentiable at 0.

- 4. For f to be of class C^1 we need $\beta > 1$, but this is not enough, we also need $\lim_{x \to 0} f'(x) = f'(0) = 0$.
 - If $\beta > 2$ then $\lim_{x\to 0} |x|^{\beta-2} = 0$, and since sin and cos are bounded, we have

$$\lim_{x \to 0} f' = \lim_{x \to 0} |x^{\beta - 2}| \left(x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right) = 0 = f'(0)$$

so that f is of class C^1 in this cases.

$$f'(x) = x^{\beta-2} \left(x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right).$$

We proceed by contradiction and assume that $\lim_{0+} f' = \ell \in \mathbb{R}$. Notice that

$$\cos\left(\frac{1}{x}\right) = x\sin\left(\frac{1}{x}\right) - x^{2-\beta}f'(x),$$

and that, since sin is bounded:

$$x \sin\left(\frac{1}{x}\right) \underset{x \to 0^+}{\longrightarrow} 0.$$

Then

$$\lim_{+\infty} \cos = \lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right) - x^{2-\beta} f'(x) = \begin{cases} -\ell & \text{if } \beta = 2\\ 0 & \text{otherwise} \end{cases}$$

which is impossible. Hence $\lim_{0+}f'$ doesn't exist, hence f is not of class C^1

We conclude that f is of class C^1 if and only if $\beta > 2$.

5. For example the function

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto \begin{cases} |x|^{3/2} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This corresponds to the case $\beta = 3/2$: $\beta > 1$ so that f is differentiable, but $\beta \leq 2$ so that f is not of class C^1 .

Exercise 4.

$$\underbrace{3}_{e^x = x \to 0} = 1 + x + \frac{x^2}{2} + o(x^2),$$

$$\sin(x)x = x - \frac{x^3}{6} + o(x^3).$$

$$e^{x} \sin(x) \underset{x \to 0}{=} \left(1 + x + \frac{x^{2}}{2} + o(x^{2})\right) \left(1 - \frac{x^{3}}{6} + o(x^{3})\right)$$
$$\underset{x \to 0}{=} x + x^{2} + \frac{x^{3}}{3} + o(x^{3}).$$

$$f(x) \underset{x \to 0}{=} x + o(x) \underset{x \to 0}{\sim} x,$$

$$f(x) - x \underset{x \to 0}{=} x^2 + o(x^2) \underset{x \to 0}{\sim} x^2,$$

$$f(x) - x - x^2 \underset{x \to 0}{=} \frac{x^3}{3} + o(x^3) \underset{x \to 0}{\sim} \frac{x^3}{3}$$

$$\frac{1}{x^3} (e^x \sin(x) - x - x^2) \underset{x \to 0}{\sim} \frac{1}{3} \xrightarrow[x \to 0]{} \frac{1}{3}$$

hence $\ell = 1/3$.

Exercise 5.



- 1. See booklet.
- 2. See lecture.