INSA | INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 – Math Test #5 – 3h00

June 22, 2023

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

The marks are given as a guide only; the final marking scheme might differ *slightly* from the marks provided here.

Exercise 1 (*2.5 points*). The two question of this exercise are independent of each other.

1. Let
$$g : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \cos(x)\cosh(x) - 1. \end{array}$$
Use a Taylor expansion to determine an equivalent of g at 0 of the form ax^b with $a, b \in \mathbb{R}^*$.

2. Let
$$f : \begin{array}{rcl} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^2 \ln\left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right). \end{array}$$
   1. Fill in the blank with the third order Taylor–Young expansion:
   $$\ln(1+h) \underset{h\to 0}{= =} \quad \cdots \quad + o(h^3)$$
   (no justifications required).
   2. Show that $f \underset{x\to+\infty}{=} \alpha x + \beta + \dfrac{\gamma}{x} + o\left(\frac{1}{x}\right)$ where $\alpha, \beta, \gamma \in \mathbb{R}^3$ you will determine.
   3. What can you deduce about the graph of f in a neighborhood of $+\infty$? (asymptote? relative position with respect to this asymptote?). Sketch a graph.

Exercise 2 (*4 points*). Let $E = \mathbb{R}^3$.
We denote (as usual) by id the identity function of $\mathbb{R}^3$, i.e., the endomorphism of E such that $\forall v \in E,\ \mathrm{id}(v) = v$.

1. Preliminary question: let φ be an endomorphism of E. Let $v \in E$ and $k \in \mathbb{R}$. Show that:
$$\varphi(v) = kv \iff v \in \mathrm{Ker}(\varphi - k\ \mathrm{id}).$$

Let f be the endomorphism of E the matrix of which in the standard basis of $\mathbb{R}^3$ is:
$$M = [f]_{\mathrm{std}} = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 2 & 0 \\ 0 & -3 & 5 \end{pmatrix}.$$

2. Let $g = f - 5\,\mathrm{id}$ and $h = f - 2\,\mathrm{id}$. Determine the matrices $M_g = [g]_{\mathrm{std}}$ and $M_h = [h]_{\mathrm{std}}$ of g and h in the standard basis.
3. a) Show that $\forall v \in E,\ (g \circ h)(v) = 0_E$.
   b) Deduce, without any further computations that $\mathrm{Im}\, h \subset \mathrm{Ker}\, g$.
4. Give, without any justifications, a basis $\mathscr{B}_g$ of $\mathrm{Im}\, g$ and a basis $\mathscr{B}_h$ of $\mathrm{Im}\, h$.
5. Show that $\mathrm{Im}\, h = \mathrm{Ker}\, g$.
   It can be proven that $\mathrm{Im}\, g = \mathrm{Ker}\, h$ (and you may use this result without any justifications).
6. Show that $\mathrm{Im}\, g$ and $\mathrm{Im}\, h$ are complementary subspaces of E.
7. Deduce a basis $\mathscr{C}$ of E such that the matrix $D = [f]_{\mathscr{C}}$ of f in $\mathscr{C}$ is diagonal, and explicit this matrix D.

Exercise 3 (*5 points*). Let

$$\begin{aligned} f : \mathbb{R}^* &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1-e^{-x}}{x}. \end{aligned}$$

1. Study at 0.

   a) Give the second order Taylor–Young expansion of f at 0, i.e., with $o(x^2)$.

   b) Justify that f possesses a continuous extension at 0.
   In the sequel we still denote by $\tilde{f}$ this extension by continuity, and by C the graph of $\tilde{f}$.

   c) Justify that $\tilde{f}$ is differentiable at 0.

   d) Give the tangent line Δ to C at $(0, \tilde{f}(0))$, as well as the relative position of C with respect to Δ in a neighborhood of 0. Sketch a graph.

   e) It can be shown that $\tilde{f}$ is twice differentiable at 0. With this information and no further computations, determine the value of $\tilde{f}''(0)$.

2. Study at $\pm\infty$.

   a) Give a simple equivalent of f at $+\infty$ as well as the limit of f at $+\infty$.

   b) Same question at $-\infty$.

3. Variations of f:

   a) Let $x \in \mathbb{R}^*$. Compute $f'(x)$.

   b) Show using the Mean Value Theorem that for all $u \in \mathbb{R}^*$, $u < e^u - 1 < ue^u$.

   c) Deduce the variations of f.

4. a) Let $x \in \mathbb{R}^*$. Show that there exists $c_x \in (0, x)$ such that

   $$e^{-x} = 1 - x + \frac{x^2}{2} e^{-c_x}.$$

   b) Deduce an expression of $f(x)$ in terms of c_x.

   c) Deduce the *global* position of the graph of f with respect to Δ (where Δ is the tangent line established in Question 1.d).

Exercise 4 (*3 points*). Let E be a vector space of dimension 3 with basis $\mathscr{B} = (e_1, e_2, e_3)$.
Define:

$$u_1 = (1, 1, -3), \qquad u_2 = (1, -1, 0), \qquad u_3 = (1, 0, -1)$$

and set $\mathscr{C} = (u_1, u_2, u_3)$.
Let $f \in L(E)$ be the endomorphism of E the matrix of which, in the basis $\mathscr{B}$ is:

$$A = [f]_{\mathscr{B}} = \begin{pmatrix} 4 & 3 & 2 \\ 0 & 1 & 0 \\ -3 & -3 & -1 \end{pmatrix}.$$

1. Express the change of basis matrix $P = [\mathscr{C}]_{\mathscr{B}}$, and use it to prove that $\mathscr{C}$ is a basis of E.
2. Using the change of basis formula, determine the matrix $D = [f]_{\mathscr{C}}$ of f in the basis $\mathscr{C}$ (you should find a diagonal matrix).
3. Deduce, for $n \in \mathbb{N}$, an explicit expression for D^n and A^n

Exercise 5 (*5.5 points*). Let

$$\begin{array}{rccc} \varphi : & \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^3 \\ & P & \longmapsto & (P(1), P'(1), P''(1)). \end{array}$$

Let $\mathscr{B} = (1, X, X^2)$ the standard basis of $\mathbb{R}_2[X]$ and let $\mathscr{C} = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$.

1. a) Determine the matrix $A = [\varphi]_{\mathscr{B},\mathscr{C}}$ in the bases $\mathscr{B}, \mathscr{C}$.

   b) Justify that φ is a bijection.

2. You're given that the family $\mathscr{B}' = (1, X-1, (X-1)^2)$ is a basis of $\mathbb{R}_2[X]$.

   a) Without using the change of basis formula, compute the matrix $N = [\varphi]_{\mathscr{B}',\mathscr{C}}$ of φ in the bases $\mathscr{B}', \mathscr{C}$.

   b) Let $u \in (a, b, c) \in \mathbb{R}^3$. Compute $\varphi^{-1}(u)$. (You don't need to give the expanded form).

   c) Let $P \in \mathbb{R}_2[X]$. Deduce an expression of $(\varphi^{-1} \circ \varphi)(P)$ using $\mathscr{B}'$. Obviously, $P = (\varphi^{-1} \circ \varphi)(P) = P$. What formula does this illustrate?

3. a) Give the change of basis matrix $[\mathscr{B}']_{\mathscr{B}}$.

   b) Give a relation between N, A and R.

4. From now on we consider the vector space $E = \mathscr{D}^2(\mathbb{R})$ that consists of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are twice differentiable. We also define

$$\begin{array}{rccc} \psi : & E & \longrightarrow & \mathbb{R}^3 \\ & f & \longmapsto & (f(1), f'(1), f''(1)). \end{array}$$

   a) Determine whether ψ is injective.

   b) Is the mapping ψ surjective?

   c) Let $G = \{f \in E \mid f(x) \underset{x\to 1}{=} o((x-1)^2)\}$. Show that $\operatorname{Ker}\psi = G$. (Detail with care).

   d) Deduce the set $\psi^{[-1]}((0,1,1)) = \{f \in E \mid \psi(f) = (0,1,0)\}$.