INSA | INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #1 — 1h45

October 28, 2022

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1.

1. a) Recall (without any justifications) the Binomial Theorem.

   b) Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$. Recall (without any justifications) the value of the sum:

$$\sum_{k=0}^{n} q^k.$$

   You will separate the case $q \neq 1$ and $q = 1$.

2. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that the following expression is well defined, and compute its value:

$$S = \sum_{k=0}^{n} \frac{1}{\sum_{j=0}^{k} \binom{k}{j} x^{2j}}$$

Exercise 2. You are given (and you may use this property without any justifications):

$$(*) \qquad \forall x \in \mathbb{R}, \ |\sin(x)| \leq |x|.$$

1. Use $(*)$ to show that $\pi \geq 3$. *Hint: use a value of* $\pi/6$.
2. Use the fact that $\pi/12 = \pi/3 - \pi/4$ to determine the value of $\sin(\pi/12)$, $\cos(\pi/12)$ and of $\tan(\pi/12)$.
3. Deduce that $\pi \geq 3\sqrt{2}\left(\sqrt{(3)} - 1\right)$. *Note: an approximation of this value is* $3.1058\ldots$.

Exercise 3. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$k \longmapsto \begin{cases} t & \text{if } t > 0 \\ 1 & \text{if } t = 0 \\ -\dfrac{1}{t} & \text{if } t < 0 \end{cases}$$

You're given that f is well defined.

1. Sketch the graph of f.
2. What is the range of f? (no justifications required).
3. Is the function f surjective? injective? bijective?
4. Determine graphically the following sets (no justifications required):

$$f(\mathbb{R}_+), \quad f(\mathbb{R}_+^*), \quad f(\,[0,1)\,), \quad f(\,(-1,0]\,),$$
$$f^{[-1]}(\mathbb{R}_-), \quad f^{[-1]}(\,[1,+\infty)\,), \quad f^{[-1]}(\,(-\infty,1]\,), \quad f^{[-1]}(\,[1,2)\,).$$

Exercise 4. Let A be a non empty set, and let $f : A \longrightarrow A$ be a function.

1. Explain why $f \circ f$ is well defined.
2. Let $g : A \longrightarrow A$ be a bijection such that $f \circ f = g$. Show that f is a bijection.

Exercise 5. Define

$$\begin{array}{rccc} f : & \mathbb{R} \setminus \{-1\} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{2x^2 + 2x + 1}{1 + x}. \end{array}$$

1. Determine the largest subset D_g of $\mathbb{R}$ so that the function

$$\begin{array}{rccc} g : & D_g & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x-1) + 2 \end{array}$$

   is well defined, and show that g is odd.

2. Show that g is increasing on $\left[1/\sqrt{2}, +\infty\right)$ and decreasing on $\left(0, 1/\sqrt{2}\right]$.

   Hint. You are given that

$$\forall x, y \in D_g,\ g(x) - g(y) = \frac{(2xy-1)(x-y)}{xy},$$

   and you may use this fact without any justifications.

3. Sketch the graph of g and the graph of f, explaining how you obtain the graph of f from that of g.

Exercise 6. Let $A = (0, 1)$. Are the following propositions true or false? (provide a proof or a counterexample to justify your claim).

$$\begin{array}{ll} (P_1) & \forall x \in A,\ \forall y \in A,\ x + y \in A \\ (P_2) & \forall x \in A,\ \exists y \in A,\ x + y \in A \\ (P_3) & \exists x \in A,\ \forall y \in A,\ x + y \in A. \end{array}$$