INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #2 — 1h45

December 9, 2022

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Define:

$$P : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 2x^5 - 7x^4 + 10x^3 - 5x^2 - 2x + 2 \end{array}.$$

1. You're given that $1+i$ and 1 are roots of P. Determine all the roots of P and their multiplicities.
2. Give the factored form of P in $\mathbb{R}$ and in $\mathbb{C}$.

Exercise 2. The two questions of this exercise are independent of each other.

1. Let $x \in \mathbb{R}$.
   a) Check that the expression
   $$A = \frac{1+\tanh(x)}{1-\tanh(x)}$$
   is well-defined, and simplify A as much as you can (*hint: use the exponential*).
   b) Deduce that:
   $$\forall n \in \mathbb{N}, \quad \left(\frac{1+\tanh(x)}{1-\tanh(x)}\right)^n = \frac{1+\tanh(nx)}{1-\tanh(nx)}.$$
2. a) Determine the maximal subset J of $\mathbb{R}$ such that for all $x \in J$ the expression
   $$B = \operatorname{arccosh}\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right)$$
   is well-defined.
   b) For $x \in J$, simplify B as much as you can.

Exercise 3. Let $E = \mathbb{R}[X]$ be the vector space of all polynomials with real coefficients and indeterminate X. Let

$$P_1 = \{P \in E \mid P(0) + P(1) = 0\},$$

and

$$P_2 = \{P \in E \mid \deg(P) \leq 2\} = \operatorname{Span}\{1, X, X^2\}$$

You're given that P_2 is a subspace of E.

1. Show that P_1 is a subspace of E.
2. Is $P_1 \cap P_2$ a subspace of E?
3. Show that there exists $p, q \in E$ such that $P_1 \cap P_2 = \operatorname{Span}\{p, q\}$ (and determine such p and q).

Exercise 4.

1. Let $m \in \mathbb{R}$. What is the rank of the following system?

$$(S) \qquad \begin{cases} x+\ y+\ z=m \\ x+2y+3z=1 \\ 2x+5y+8z=2 \end{cases}$$

   Solve System (S).

2. Let $E = \mathbb{R}^3$, let $m \in \mathbb{R}$ and define:

$$u = (1,1,2), \qquad v = (1,2,5), \qquad w = (1,3,8), \qquad a = (m,1,2).$$

   Note that some answers to the following questions might depend on the value of m.

   a) Do we have $a \in \text{Span}\{u, v, w\}$?

   b) Is the family (u, v, w) a generating family of $\mathbb{R}^3$?

   c) i) What is the rank of the following linear system?

$$(S') \qquad \begin{cases} x+\ y+\ z=0 \\ x+2y+3z=0 \\ 2x+5y+8z=0 \end{cases}$$

   ii) Without solving System (S'), determine the number of solutions of System (S').

   iii) Is the family (u, v, w) an independent family?

Exercise 5. Let $E = \mathbb{R}^4$ and define:

$$a = (1,1,1,0), \qquad b = (1,1,0,1), \qquad c = (1,-1,0,0), \qquad d = (1,1,1,1),$$

$$F = \text{Span}\{a, b\}, \qquad G = \text{Span}\{c, d\}.$$

1. Find all the real numbers $\alpha, \beta, \gamma, \delta$ such that

$$\alpha a + \beta b = \gamma c + \delta d.$$

   Deduce that $F \cap G = \{0_E\}$.

2. We now show that $E = F + G$.

   a) Let $u = (x, y, z, t) \in E$. Show that there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$u = \alpha a + \beta b + \gamma c + \delta d.$$

   Can we conclude that the family (a, b, c, d) is a generating family of E?

   b) Use the result of the previous question to find $u_F \in F$ and $u_G \in G$ such that $u = u_F + u_G$. How many possibilities are there for u_F and u_G?

   c) Is this enough to conclude that $E = F + G$?

3. Do we have $E = F \oplus G$?

We recall the definition of an independent family and a generating family (these notions are used in Exercises 4 and 5): Let E be a vector space over $\mathbb{K}$ and let $\mathcal{F} = (u_1, \ldots, u_n)$ be a family of n vectors of E (with $n \geq 1$).

- We say that $\mathcal{F}$ is *independent* if:

$$\forall \lambda_1, \ldots, \lambda_n \in \mathbb{K}, \ (\lambda_1 u_1 + \cdots + \lambda_n u_n = 0_E \implies \lambda_1 = \cdots = \lambda_n = 0)$$

- We say that $\mathcal{F}$ is a *generating family of E* if $\text{Span}\{u_1, \ldots, u_n\} = E$.