SCAN 1 — Math Test #3 — 1h30

March 17, 2023

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The two questions of this exercise are independent from each other.

1. Let $A = (0, 1]$. Determine (in case of existence) the values of $\sup A$, $\inf A$, $\min A$ and $\max A$. No justifications required.
2. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} x^2 & \text{if } x > 0 \\ \cosh(x) & \text{if } x \leq 0. \end{cases}$$

   Sketch the graph of f and determine (in case of existence) the values of $\sup f$, $\inf f$, $\max f$ and $\min f$.

Exercise 2. The questions of this exercise are independent from each other.

1. Determine the value of the following limits (if a limit doesn't exist write "DNE"). Justify your answer (in the case the limit is not a limit to be known by heart).

$$(1)\ \lim_{x\to 0} \frac{\sin(x)}{x}, \qquad (2)\ \lim_{x\to 0} \frac{e^x - 1}{\ln(1+x)}, \qquad (3)\ \lim_{x\to 0} \frac{\sin(x^2)}{\cos(x) - 1}.$$

2. Let $\alpha \in \mathbb{R}$. Determine the value of the following limit (or "DNE") if the limit doesn't exist.

$$\lim_{x\to+\infty} \frac{x^\alpha + 1}{x^\alpha + \ln(x)}$$

3. Determine $\lim\limits_{+\infty} \tanh$. Do we have $\sinh \underset{+\infty}{\sim} \cosh$?

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall x, y \in \mathbb{R},\ |f(x) - f(y)| \leq \frac{1}{2}|x - y|.$$

1. Let $x, y \in \mathbb{R}$ such that $f(x) = x$ and $f(y) = y$. Show that $x = y$.
2. From now on we assume that there exists $a \in \mathbb{R}$ such that $f(a) = a$ (such an a is unique from the previous question). Show that $\lim\limits_{a} f = a$.

Exercise 4. Let $E = \mathbb{R}_2[X]$ and define

$$P_1 = X^2 + X - 1, \qquad P_2 = X^2 - X + 1, \qquad P_3 = -X^2 + X + 1.$$

1. Show that $\mathscr{B} = (P_1, P_2, P_3)$ is a basis of E.
2. Let $P \in E$ be such that

$$[P]_{\mathscr{B}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

   Determine P (i.e., write P as a linear combination of 1, X and X^2).

3. Determine the coordinates of X^2, X and 1 in the basis $\mathscr{B}$.
4. We define

$$F_1 = \operatorname{Span}\{P_1\}, \qquad F_2 = \operatorname{Span}\{P_2, P_3\},$$

   Let $a, b, c \in \mathbb{R}$ and define $Q = aX^2 + bX + c$. Find $Q_1 \in F_1$ and $Q_2 \in F_2$ such that $Q = Q_1 + Q_2$.

5. Define

$$G = \{P \in E \mid P(0) = P(1)\}.$$

   a) Check that G is a subspace of E.
   b) Do we have $G = E$?
   c) Check that $F_2 \subset G$. Can we conclude that $F_2 = G$?
   d) Do we have $E = F_1 \oplus G$?

Exercise 5. Let $E = \mathbb{R}^3$ and $F = \mathbb{R}_2[X]$ and define the following vectors of E:

$$u_1 = (1, 0, 1), \qquad u_2 = (0, 1, 1), \qquad u_3 = (1, 1, 0).$$

1. Show that $\mathscr{B} = (u_1, u_2, u_3)$ is a basis of E.
2. Explain why there exists a unique linear map $f : E \longrightarrow F$ such that

$$f(u_1) = X, \qquad f(u_2) = 1, \qquad f(u_3) = X^2 + 1.$$

3. Determine the matrix $A = [f]_{\mathscr{B},\mathrm{std}_F}$ of f in the bases $\mathscr{B}$ and the standard basis of F.
4. Determine the matrix $B = [f]_{\mathrm{std}_E,\mathrm{std}_F}$ of f in the standard basis of E and the standard basis of F.
5. By determining the kernel and/or the image of f, determine whether f is injective, surjective, bijective.

Exercise 6. Let $E = \mathbb{R}^3$ and let $f : E \longrightarrow E$ be the linear map the matrix of which, in the standard basis of E is

$$[f]_{\mathrm{std}} = A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & -1 \\ -1 & 2 & 0 \end{pmatrix}$$

and define $f_1 = f - \mathrm{id}_E$.

1. Let $(x, y, z) \in E$. Explicit $f(x, y, z)$ and $f_1(x, y, z)$.
2. Determine a basis of $\operatorname{Ker} f_1$.
3. Show that there exists a unique $\lambda \in \mathbb{R} \setminus \{1\}$ (that you will determine) such that $f_\lambda = f - \lambda\,\mathrm{id}_E$ is not injective, and determine a basis of $\operatorname{Ker}(f - \lambda\,\mathrm{id}_E)$ in this case.